SHEET 1

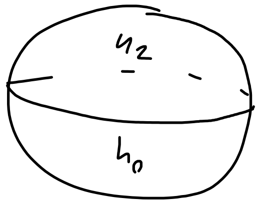
THM 0:

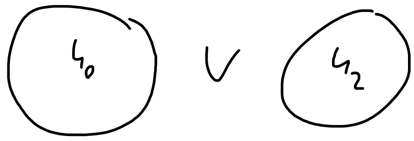
Let F = closed connected orientable surface

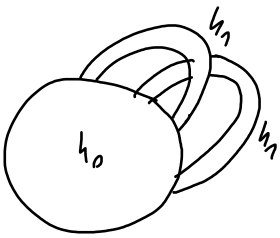
$\Rightarrow \exists !\, k \in \mathbb{N}_0 : \quad F \cong \#_k T^2$

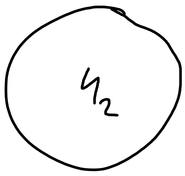
$\left(\#_0 T^2 = S^2 \quad \& \quad \#_1 T^2 = T^2 \right)$

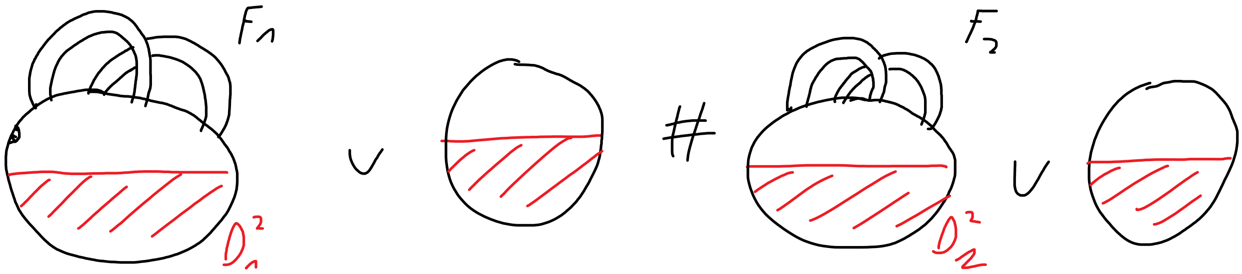
EX 1 (a)

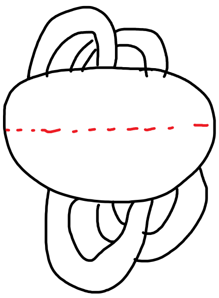
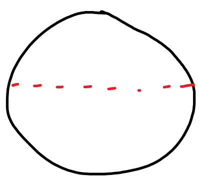
$S^2 =$ 

$=$ 

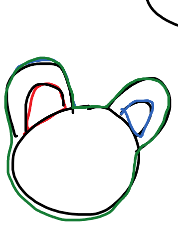
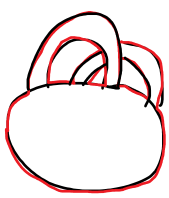
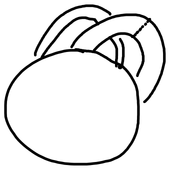
$T^2 =$ 

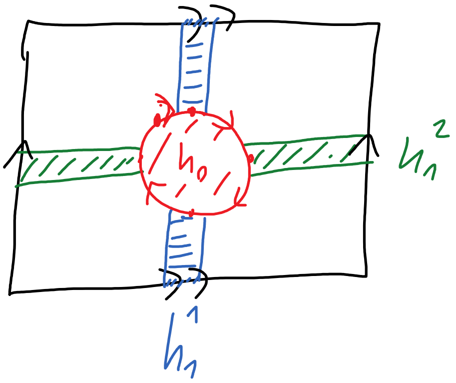
$\cup$ 

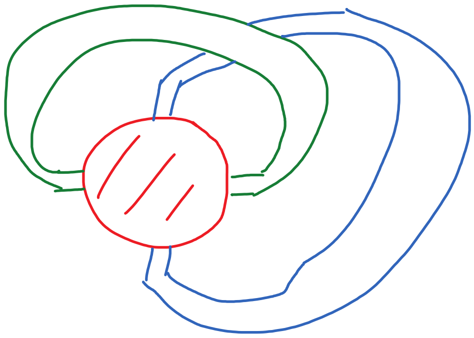
(b) 

$=$  $\cup$  ← redundant

$\square$

 $\not\cong$  $\overset{C^0}{\cong}$ 



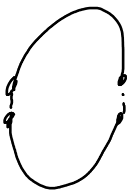


EX 2 (a) Let M^1 be a closed connected 1-mfd

CLAIM: $M \cong S^1$

Proof: M admits a hand. decomp with a unique h_0 &

" h_1



$h_1 = D^1 \times 0 = [-1,1]$

$\cong S^1$

$h_0 = 0 \times D^1 = [-1,1]$

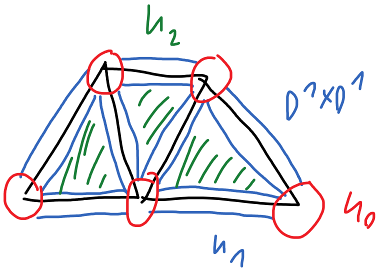
■

(b) CLAIM: $\forall F^2$ $\exists$ hand decom with a unique h_0

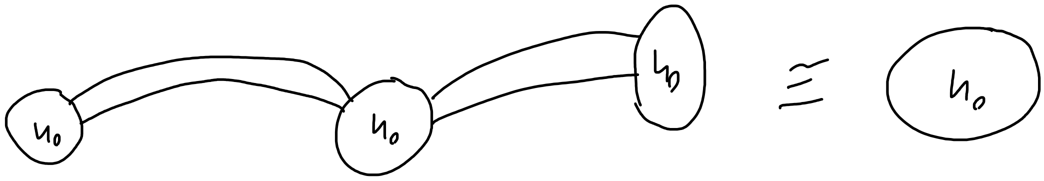
" h_2

Proof:

*



*



■

* dual handle decomp

EX 3

(a) $\phi_1, \phi_2 : \partial D^1 \times D^1 \hookrightarrow \partial F$

& $h : F \xrightarrow[\cong]{C^0} F$ s.t. $h \circ \phi_1 = \phi_2$

$\Rightarrow F \cup_{\phi_1} h_1 \overset{C^0}{\cong} F \cup_{\phi_2} h_1$

Proof:

$$\begin{tikzcd}
F_1 \arrow[dd, "\cong"] & = & F \arrow[dd, "h"] & + & h_1/\sim \arrow[dd, "\cong\ \mathrm{id}"] \\
 & & \phi_1(p) \arrow[d] \arrow[r, phantom, "\circlearrowright"] & p \arrow[l, mapsto] \arrow[d] & \\
F_2 & = & F \ \phi_2(p) & p \arrow[l, mapsto] & h_1/\sim
\end{tikzcd}$$

■

(b) ALEXANDER TRICK:

CLAIM: $\forall f : S^1 \xrightarrow[\cong]{C^0} S^1$

$\exists F : D^2 \xrightarrow{\cong} D^2$ s.t. $F|_{\partial D^2} = f$

Proof: $F : D^2 \longrightarrow D^2$

$p \longmapsto |p| \, f\left(\frac{p}{|p|}\right)$

$t \cdot x \longmapsto t \cdot f(x)$

$x \in S^1 \quad t \in [0,1]$

■

Remark: For C^∞ this is in general WRONG

But it is true for $n = 1, 2, 3, 4$

$f : \#_k S^1 \times S^2 \longrightarrow \#_k S^1 \times S^2$

$F : \natural_k S^1 \times D^3 \longrightarrow \natural_k S^1 \times D^3$

$k=0$ CERF THM

(c) $\phi_1, \phi_2 : \partial D^2 \times \{0\} = S^1 \hookrightarrow \partial F$

CLAIM: s.t. $\phi_1(S^1) = \phi_2(S^1)$

$$\Rightarrow \quad F \cup_{\phi_1} h_2 \overset{C^0}{\cong} F \cup_{\phi_2} h_2$$

Proof: $F \cup_{\phi_1} h_2 := F + \overset{D^2 \times \{0\} = D^2}{\overset{\shortparallel}{h_2}} / \sim$

$$\begin{tikzcd}
F \cup_{\phi_1} h_2 \arrow[d, dotted, "\cong"] & \phi_1(p) \arrow[d, "\cong \, id"'] \arrow[r, phantom, "\circlearrowleft"] & p \arrow[l, mapsto] \arrow[d, "\cong"] \\
F \cup_{\phi_2} h_2 & \phi_2(p) & p \arrow[l, mapsto]
\end{tikzcd}$$

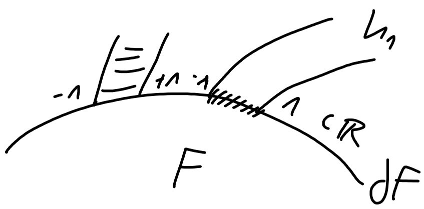
extension of $\phi_2^{-1} \circ \phi_1$ (b)

$F \cup_{\phi_2} h_2 := F + h_2 / \sim$ $\square$

EX 4: (a) CLAIM: $e : \overset{\text{a component of } \partial D^1 \times D^1}{[-1,1]} \longrightarrow \mathbb{R} \subset \partial F$ embedding $\Leftrightarrow$ e strictly monotone

Proof: * e embedding $\overset{[-1,1] \text{ compact } \& \ \mathbb{R} \text{ Hausdorff}}{\Leftrightarrow}$ e inj. & cont

$\Leftrightarrow$ strictly monotone



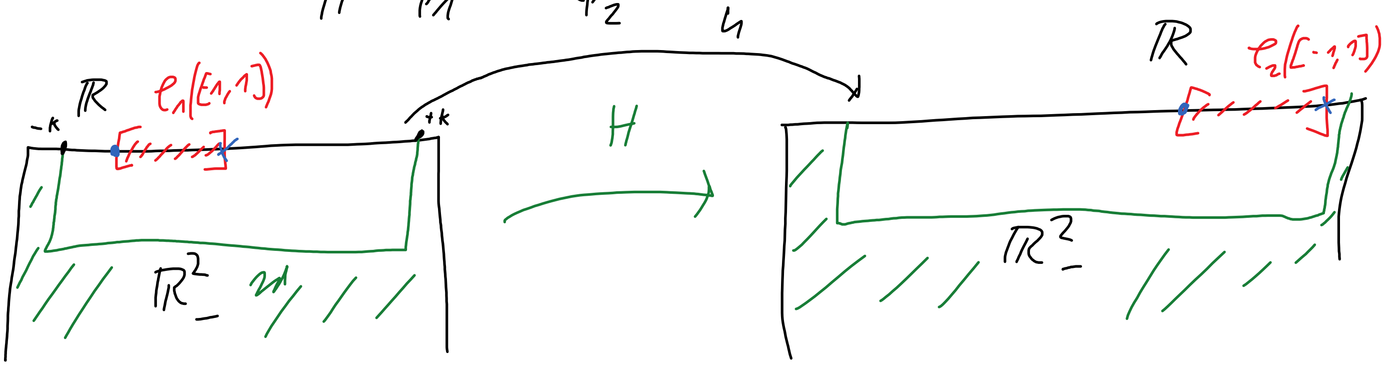
$\square$

(b) $\phi_1, \phi_2 : [-1,1] \longrightarrow \mathbb{R}$ strictly increasing

CLAIM: $\exists \; H : \mathbb{R}^2_- \overset{C^0}{\underset{\cong}{\longrightarrow}} \mathbb{R}^2_-$ s.t.

$H \equiv id$ away from a compact set

$H \circ \phi_1 = \phi_2$



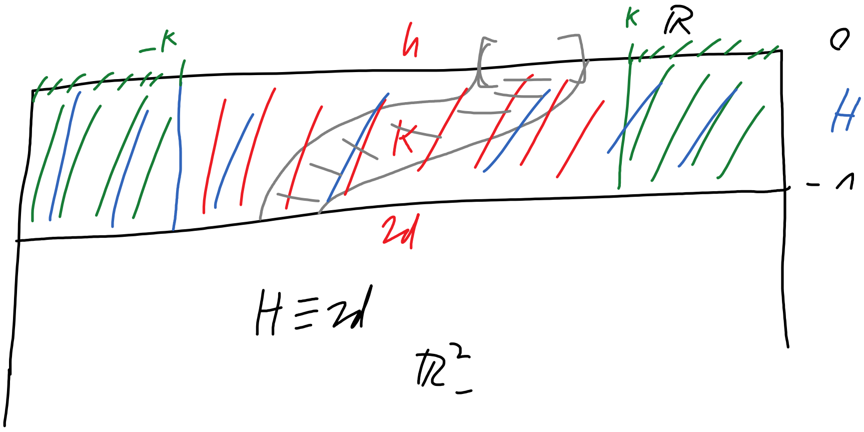
$h : \mathbb{R} \longrightarrow \mathbb{R}$ s.t. ...

$$t \longmapsto \begin{cases} \varphi_2 \circ \varphi_1^{-1}(t) & ; \ t \in \operatorname{Im}(\varphi_1) \\ \text{linear interpolation} & ; \ \text{else} \\ t & ; \ t \in \mathbb{R} \setminus [-k,k] \end{cases}$$

Let $k \in \mathbb{N}$ s.t. $\operatorname{Im}(\varphi_1) \cup \operatorname{Im}(\varphi_2) \subset [-k,k]$

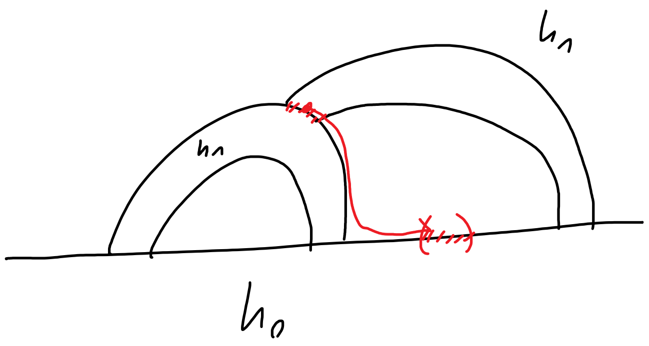
* $h \underset{H}{\simeq} \mathrm{id}$

extend h on $\mathbb{R} \times [-1,0]$ via H :



∎

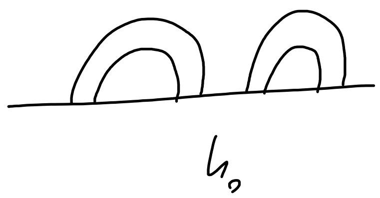
(c)



$\exists\, h$ (b)

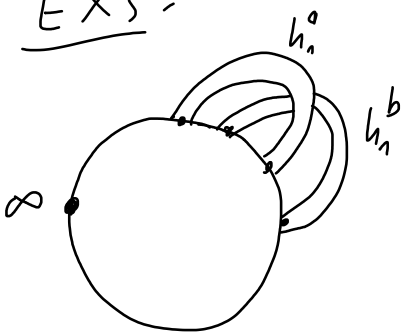
3 (a)

$=$

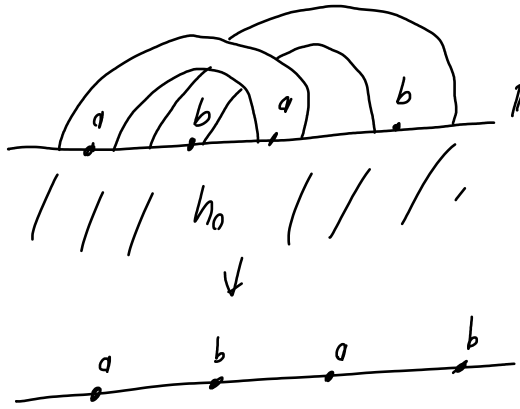


∎

EXS:

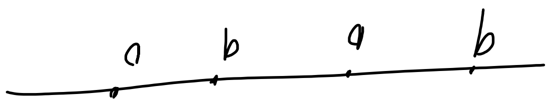


KIRBY DIAGRAMS:

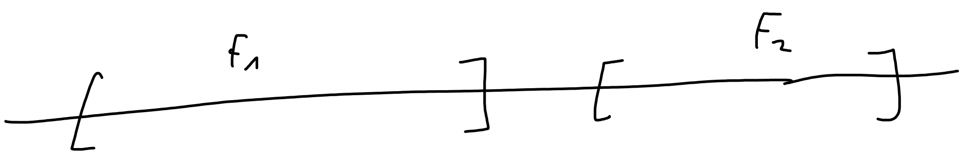


(a) $S^2 =$

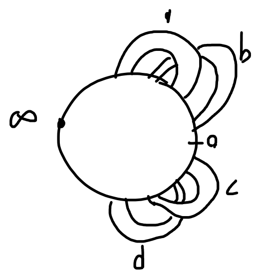
$T^2 =$



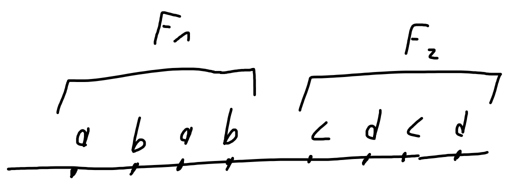
$F_1 \# F_2 =$



Example: $T^2 \# T^2 =$



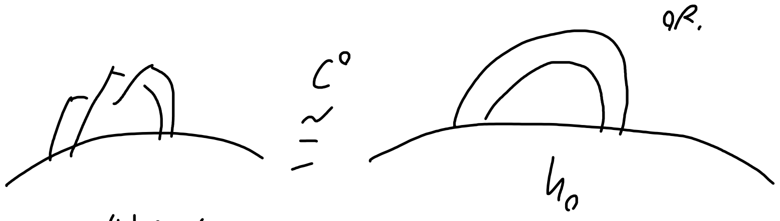
$=$



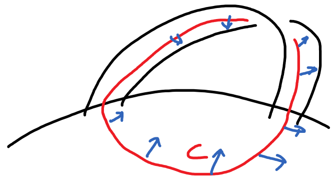
(b) CLAIM: A Kirby diagram describes a unique handle decomp. of F^2

Proof: (0) h_0 ✓

(1) F orientable $\Rightarrow$ all 1-handles are attached or. preserving



$\neq$



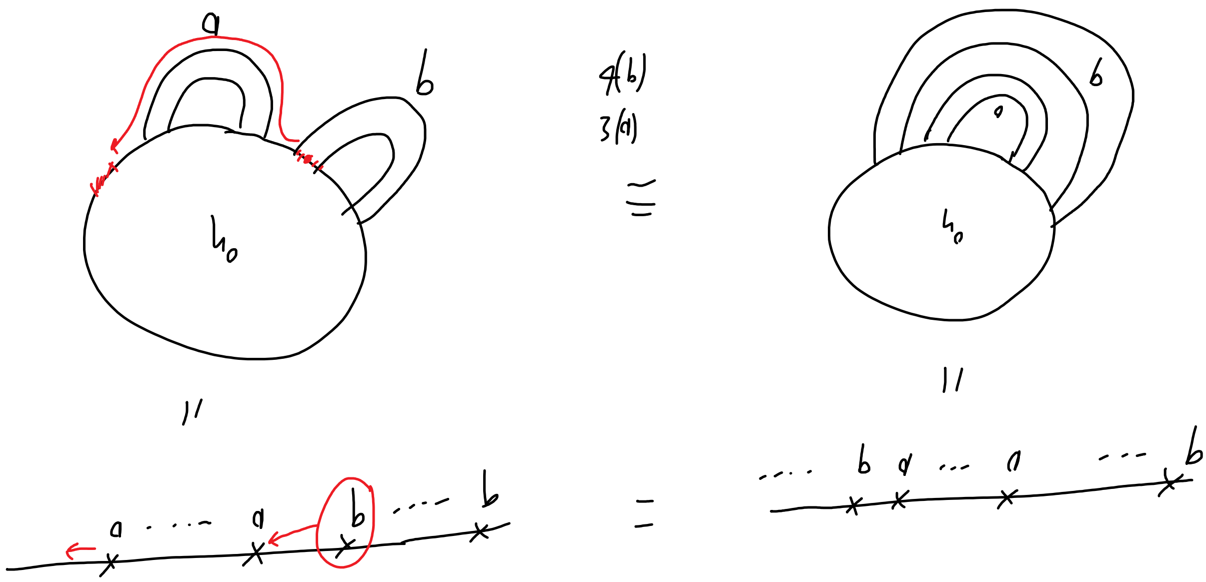
NOT ORIENTABLE

4(b) & 3(a)

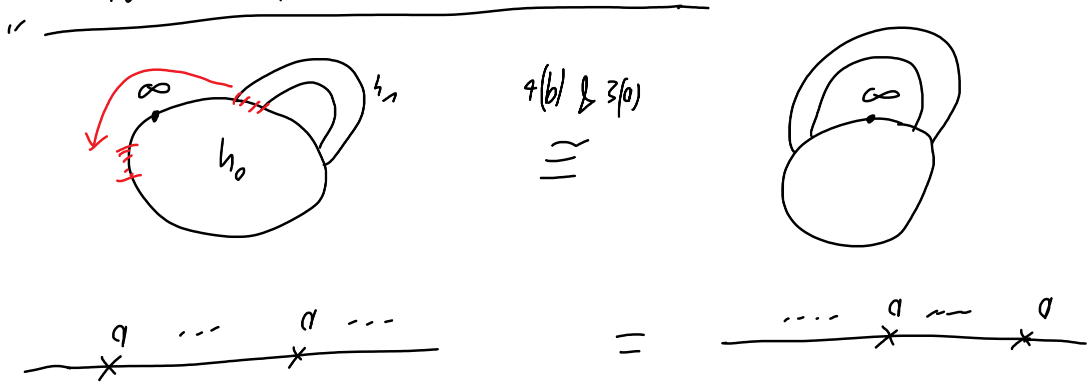
$\Rightarrow$ F_1 is det. by Kirby diag.

(2) $F = F_1 \cup h_2$ is indep. of gluing map. ∎

(c) HANDLE SLIDE:

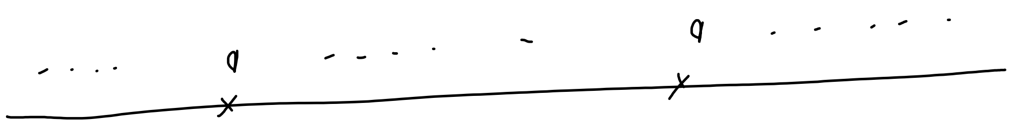


"MOVE 1-HANDLE THROUGH ∞"

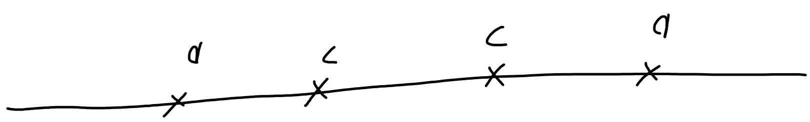


(d) CLAIM: $F \cong \#_k T^2$

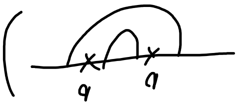
Proof: Consider h_1^a



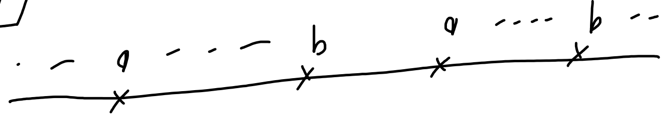
STEP 1: $\exists$ c s.t.

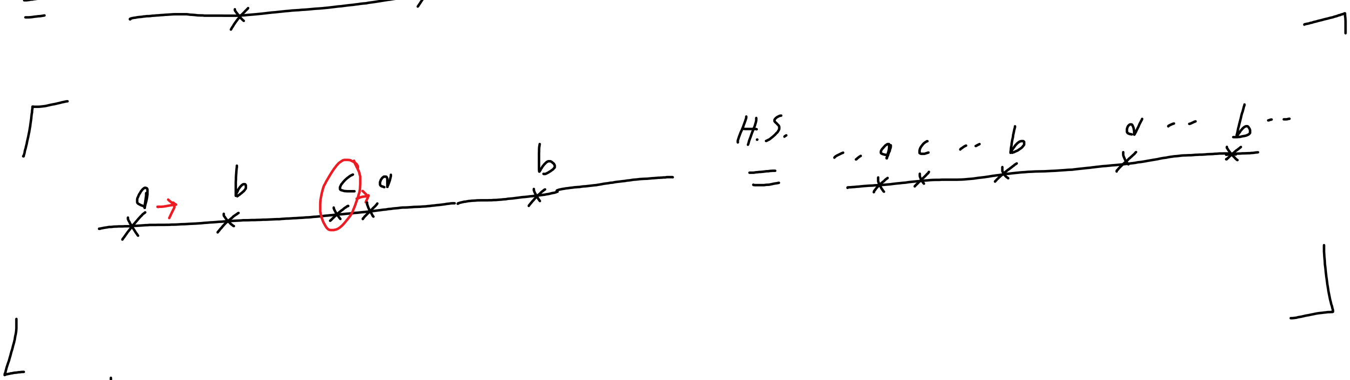


$\to$ replace a by c

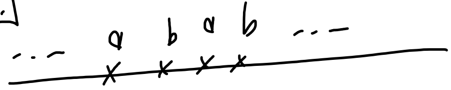
STEP 2: ( $\Rightarrow |\partial F_1| \geq 2$)

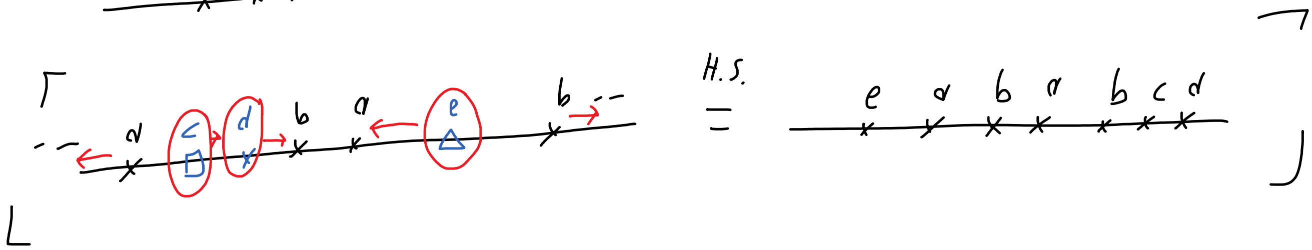
STEP 2.1 $\cong$





STEP 2.2 $\cong$





STEP 3: delete $abab$ $\hat{=}$ removing a T^2-summand of F

$\hookrightarrow$ STEP 1:

$\Rightarrow F \cong S^2 \#_k T^2$ $\blacksquare$

CLAIM: $\#_k T^2 \cong \#_l T^2 \Leftrightarrow l = k$

compute: χ, π_1 or H_1

$h(F) = \# h_0 - \# h_1 + \# h_2$ in handles of handl. decomp $(= \chi(F))$

SHEET 2 EX 1 $S^{2n+1} \subset \mathbb{C}^{n+1}$

$$\mathbb{C}P^n := S^{2n+1} \Big/ z \sim w \iff \exists \lambda \in S^1 \subset \mathbb{C} : z = \lambda w$$

we know: * $\mathbb{C}P^n$ is a smooth mfd of real dim $2n$

* $\mathbb{C}P^1 = S^2$

(a) GOAL: $h : \mathbb{C}P^n \longrightarrow \mathbb{R}$ morse

Ansatz: $\tilde{h} : S^{2n+1} \longrightarrow \mathbb{R}$

$$z \longmapsto \sum_{j=0}^{n} a_j \, |z_j|^2 \quad ; a_j \in \mathbb{R}$$

for $\lambda \in S^1$: $\tilde{h}(\lambda z) = \sum a_j \, |\lambda z_j|^2 = \tilde{h}(z)$

$\Rightarrow$ $\tilde{h}$ induces $h : \mathbb{C}P^n \longrightarrow \mathbb{R}$

charts of $\mathbb{C}P^n$: $U_k = \{ z_k \neq 0 \}$

$$\varphi_k : U_k \longrightarrow \mathbb{C}^n$$

$$[z] \longmapsto \frac{\bar{z}_k}{|z_k|} \left(z_0, \dots, \hat{z}_k, \dots, z_n \right)$$

$$\varphi_k^{-1} : (u_1, \dots, u_n) \longmapsto \left(z_0 = u_1, \dots, z_{k-1} = u_k, \; z_k = \sqrt{1 - \sum_{\substack{j=0 \\ j \neq k}}^{n} |z_j|^2}, \; z_{k+1} = u_{k+1}, \dots, z_n = u_n \right)$$

Consider $h \circ \varphi_k^{-1}$: $\underset{\mathbb{C}^n}{\overset{\cup}{U}} \longmapsto \sum_{j=0}^{n} a_j \, |z_j|^2$

$$= \sum_{j=0}^{k-1} a_j \, |u_{j+1}|^2 + a_k \Big(1 - \sum_{\substack{j=1 \\ j \neq k}}^{n} |u_j|^2 \Big) + \sum_{j=k+1}^{n} a_j \, |u_j|^2$$

$$= a_k + \sum_{j=1}^{k-1} (a_{j-1} - a_k) \, |u_j|^2 + \sum_{j=k+1}^{n} (a_j - a_k) \, |u_j|^2$$

Crit points $[0:\dots:0:1:0:\dots:0]$
$\uparrow$ k-th pos

If $a_k \neq a_j \Rightarrow$ h morse with $n+1$ crit pts of index $0,2,4,\dots,2n$

(b) $\mathbb{C}P^n = \mathbb{C}^{n+1}\setminus\{0\} / z \sim w \Leftrightarrow \exists \lambda \in \mathbb{C}: z = \lambda w$

Consider $\psi_k : \mathbb{C}^n \longrightarrow \mathbb{C}P^n$

$(u_1,\dots,u_n) \longmapsto [u_1 : \dots : u_k : 1 : \dots : u_n]$

$h_{2k} := \psi_k(\underbrace{D^2 \times \dots \times D^2}_{n\text{-times}}) \subset \mathbb{C}P^n$

CLAIM: h_{2k} is a $2k$-handle s.t. $\mathbb{C}P^n = \bigcup_{k=0}^{n} h_{2k}$

$\ulcorner$ $p \in h_{2k} \Leftrightarrow |z_k| = 1$ $\urcorner$

& $p \in \mathring{h}_{2k} \Leftrightarrow |z_j| < 1 \quad \forall j \neq k$

$\Rightarrow \bigcup h_{2k} = \mathbb{C}P^n$ & $h_{2k} \cap h_{2j} \subset \partial h_{2k}$
$\subset \partial h_{2j}$

Moreover: $\left(\bigcup_{j<k} h_{2j}\right) \cap h_{2k} = \psi_k\left(\partial(\underbrace{D^2 \times \dots \times D^2}_{k\text{-times}}) \times \underbrace{D^2 \times \dots \times D^2}_{n-k}\right)$

$\Rightarrow h_{2k}$ is a $2k$-handle attached to $\bigcup_{j<k} h_{2j}$

$\llcorner$ $\lrcorner$

$$\mathbb{C}P^2 = h_0 \cup h_2 \cup h_4$$

$$p \in h_0 \cap h_2 \quad \Rightarrow \quad p = \varphi_0(w_1, w_2) = [1 : w_1 : w_2]$$

$$p = \varphi_1(z_1, z_2) = [z_1 : 1 : z_2] = [1 : z_1^{-1} : z_1^{-1} z_2]$$

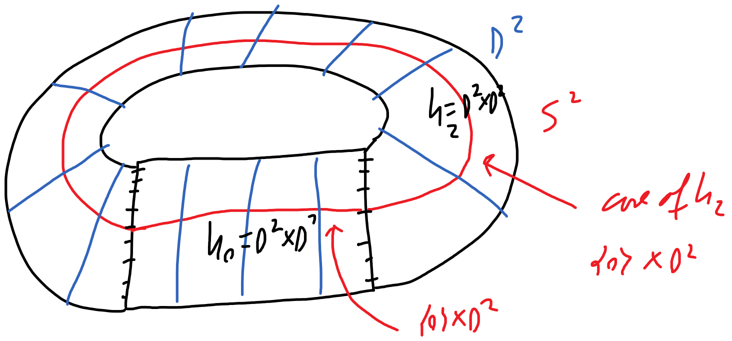
$$\Rightarrow \quad w_1 = z_1^{-1} \qquad w_2 = z_2 z_1^{-1}$$

$\Rightarrow$ attaching map of h_2

$$\varphi : \partial D^2 \times D^2 \longrightarrow \partial D^2 \times D^2 \subset \partial h_0$$

$$(z_1, z_2) \longmapsto (z_1^{-1}, z_2 z_1^{-1})$$

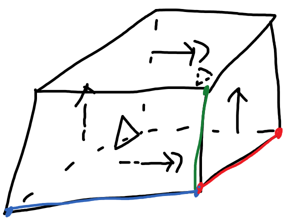
$$(e^{i\theta}, z_2) \longmapsto (e^{-i\theta}, e^{-i\theta} z_2)$$



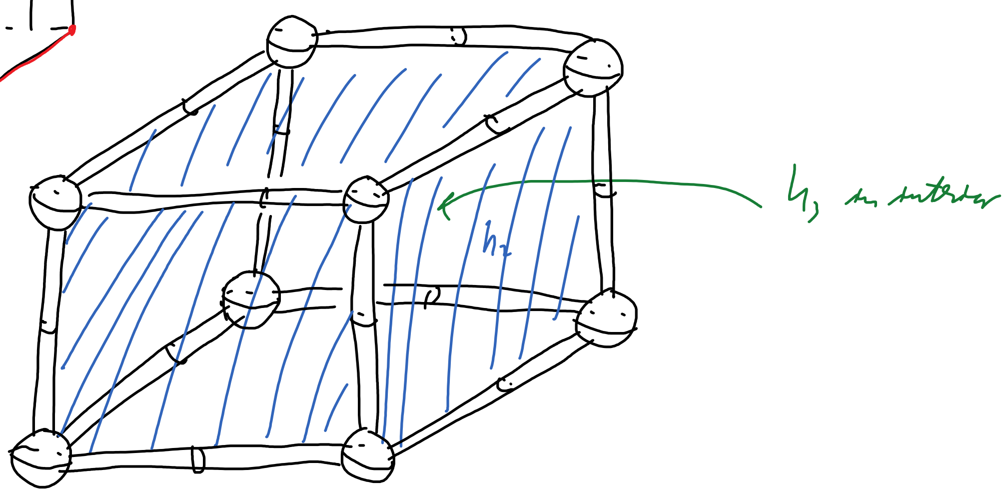
$\Rightarrow \mathbb{C}P^2 \setminus h_4$ is a D^2-bundle over S^2 ∎

$\boxed{\text{Ex 3}}$ $T^3 = S^1 \times S^1 \times S^1$

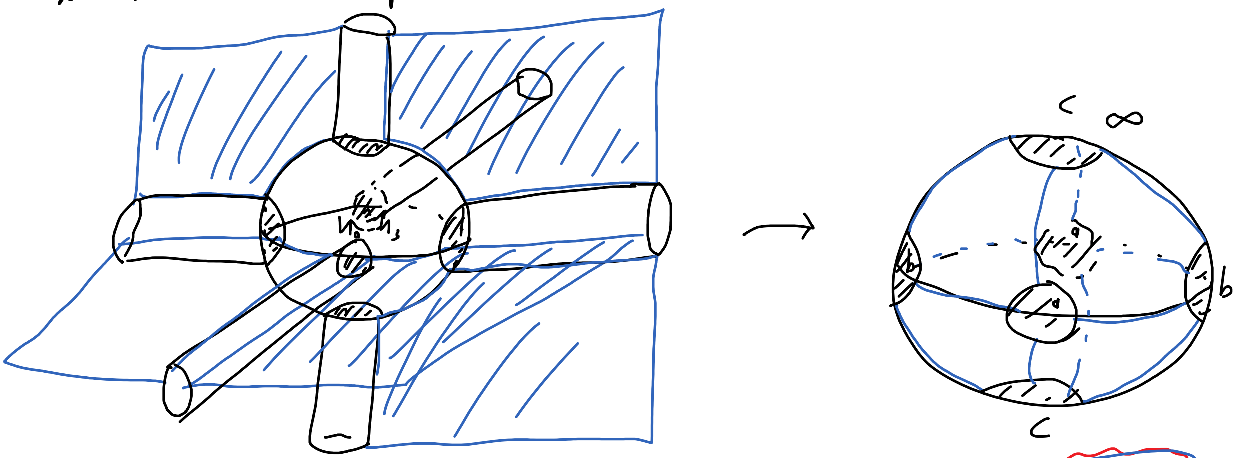
(a)

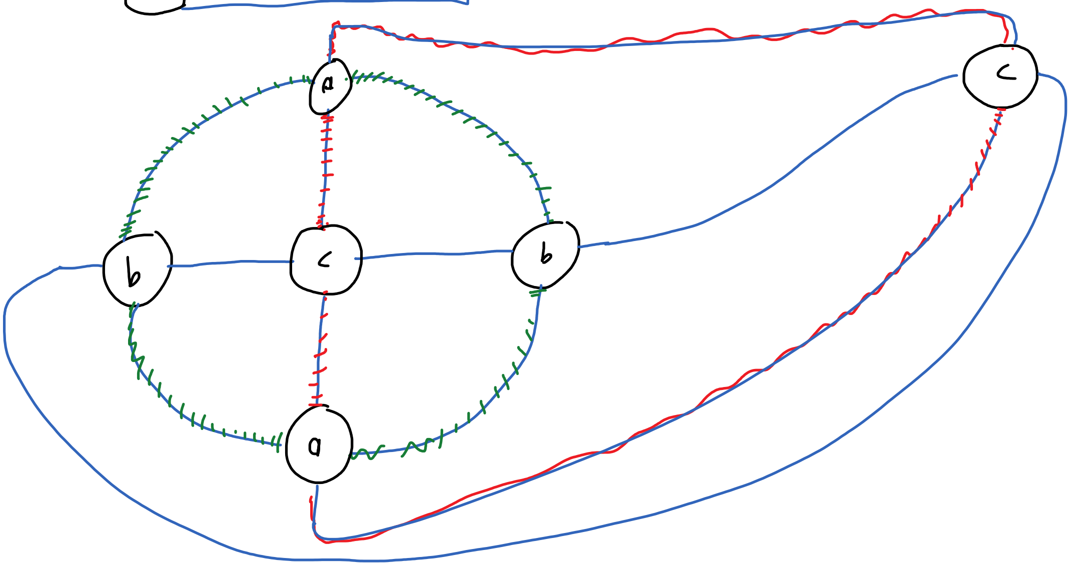


(b)



dual handle decomp





(c) CLAIM. $g(T^3) = 3$

$$g(M^3) := \{ \; g(\Sigma) \; | \; \Sigma \text{ is a Heegaard surface for a Heeg. spl. of } M \}$$

$$H_1(T^3) = \mathbb{Z}^3 \quad \Rightarrow \quad g(T^3) \geq 3$$

[Generators of $H_1(M)$ are 1-handles

$\Rightarrow$ # 1-handles $\geq \operatorname{rk}(H_1(M))$

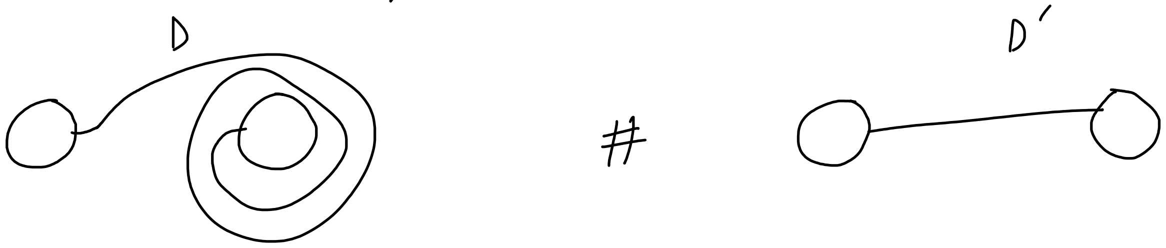
$\Rightarrow \quad g(M) \geq \operatorname{rk}(H_1(M))$]

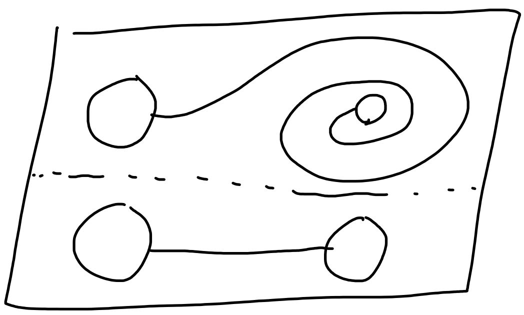
$g(T^3) \leq 3$ (c)

(e) Let M & M' 3-mfds

CLAIM: $g(M \# M') \leq g(M) + g(M')$

⌜ let D & D' be Heeg. diag of M & M' ⌝





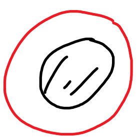
is a Heeg. diag of $M \# M'$ ⌟

Remark: HAKEN: $g(M_1 \# M_2) = g(M_1) + g(M_2)$

COR: $M = M_1 \# \dots \# M_k$ & M_i prime

(f) $g(S^1 \times S^2) = 1$

⌜

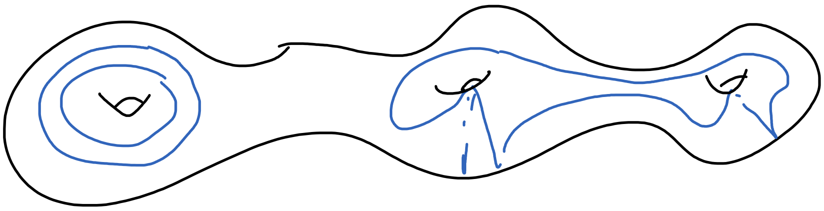
*   $= S^1 \times S^2 \Rightarrow g(S^1 \times S^2) \leq 1$ ⌝

* $H_1(S^1 \times S^2) = \mathbb{Z} \Rightarrow g(S^1 \times S^2) \geq 1$ ⌟

$g(\#_k S^1 \times S^2) = k$

BONUS:

Let $\Sigma_g = \partial M_1$



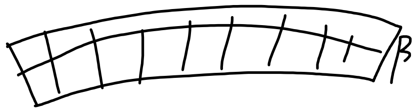
Let $\beta_1 \cup \dots \cup \beta_k$ be disjoint s.c.c. on Σ_g

$\Rightarrow$ we can attach 2-handles along β_i to M_2

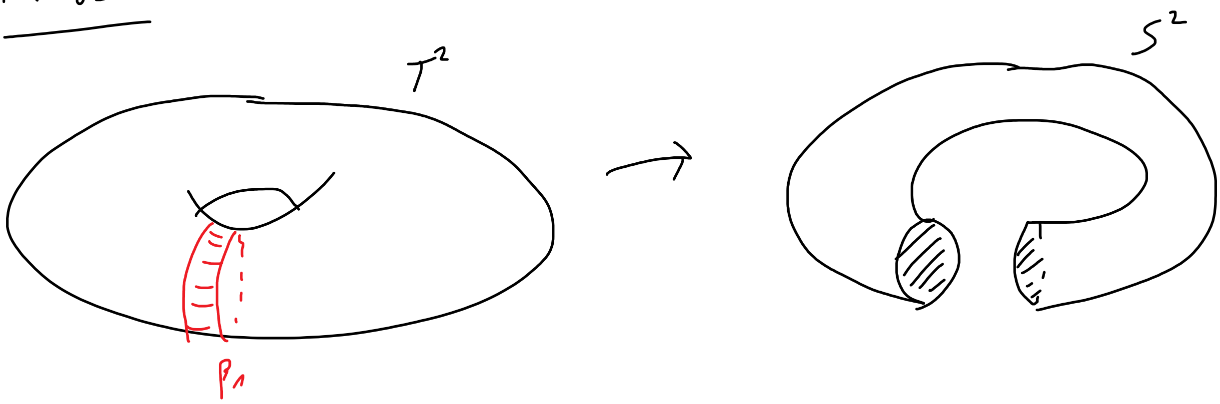
* For a Heeg. splitting we want $k = g$ & $\partial M_2 = S^2$ s.t. we can uniquely attach a 3-handle to get M.

* Attaching 2-handle to M_1 along $\beta_i \hat{=}$ surgery on Σ_g along β_i i.e.

REMOVE: $\beta_i \times D^1$



RE-GLUE: $D^2 \times S^0$



$\chi\left(\Sigma_g \setminus \beta_i \times D^1 \quad \cup D^2 \times S^0 \right)$

$= \underbrace{\chi\left(\Sigma_g \setminus \beta_i \times D^1 \right)}_{\chi(\Sigma_g)} + \underbrace{\chi\left(D^2 \times S^0 \right)}_{=2} - \underbrace{\chi\left(\beta_i \times S^0 \right)}_{=0}$

$= \quad 2 - 2g \qquad + 2$

$\Rightarrow \chi(\partial M_2) = 2 - 2g \quad + 2(\#\text{ 2-handles}) = 2 - 2g + 2g = 2$

class of surfaces
$\Rightarrow \qquad \partial M_2 = S^2 \quad \Leftrightarrow \quad \partial M_2$ is connected

we have shown:

THM:

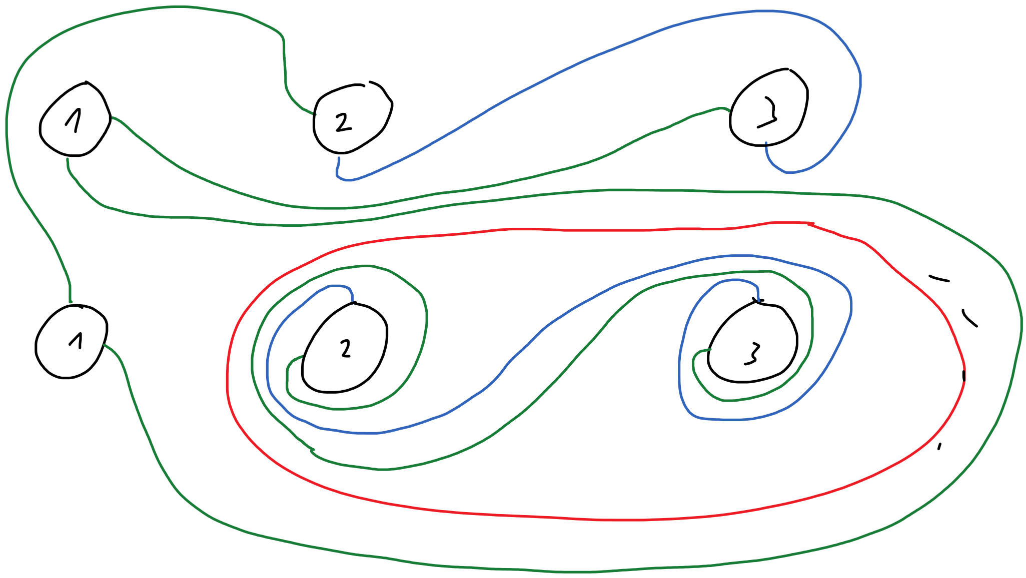
$(\Sigma_g; \beta_1, \dots, \beta_g)$ is a Heeg. diag. of a closed 3-mfd $:\Leftrightarrow$

β_i are disjoint s.c.c.

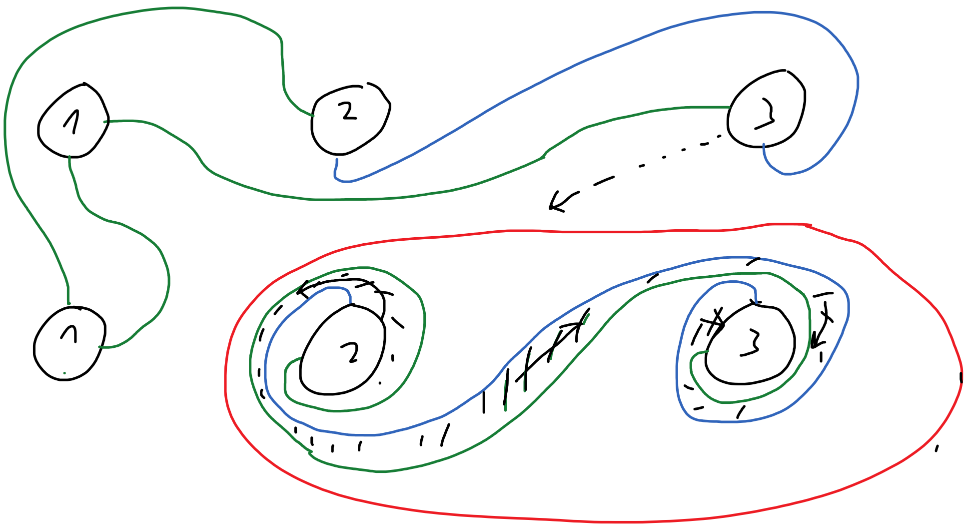
& $\Sigma_g \setminus (\beta_1 \cup \dots \cup \beta_g)$ is connected

$\blacksquare$

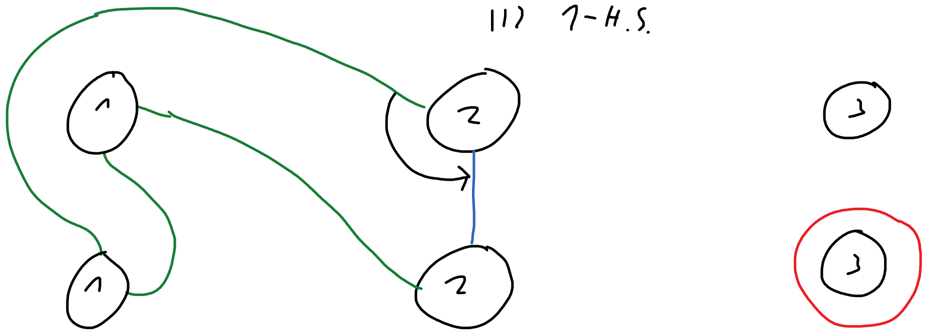
Ex 4:



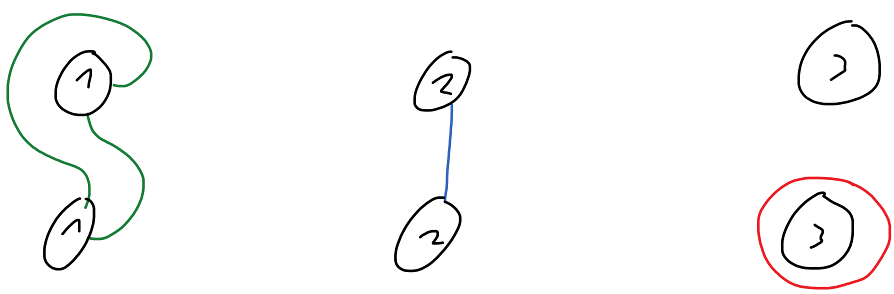
$\cong$ 2.H.S.



$\cong$ 1-H.S.



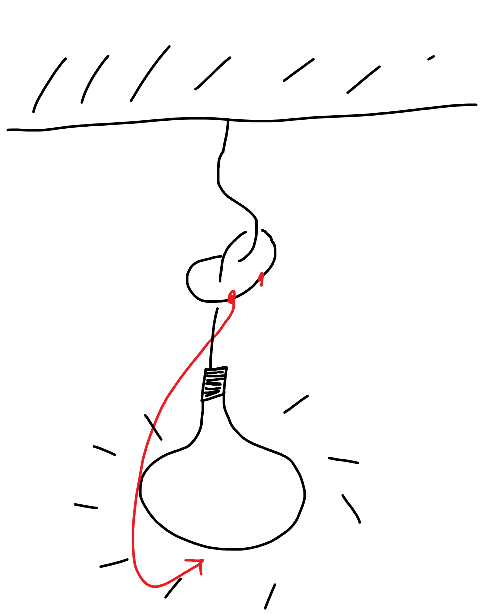
$\cong$ 2-H.S.

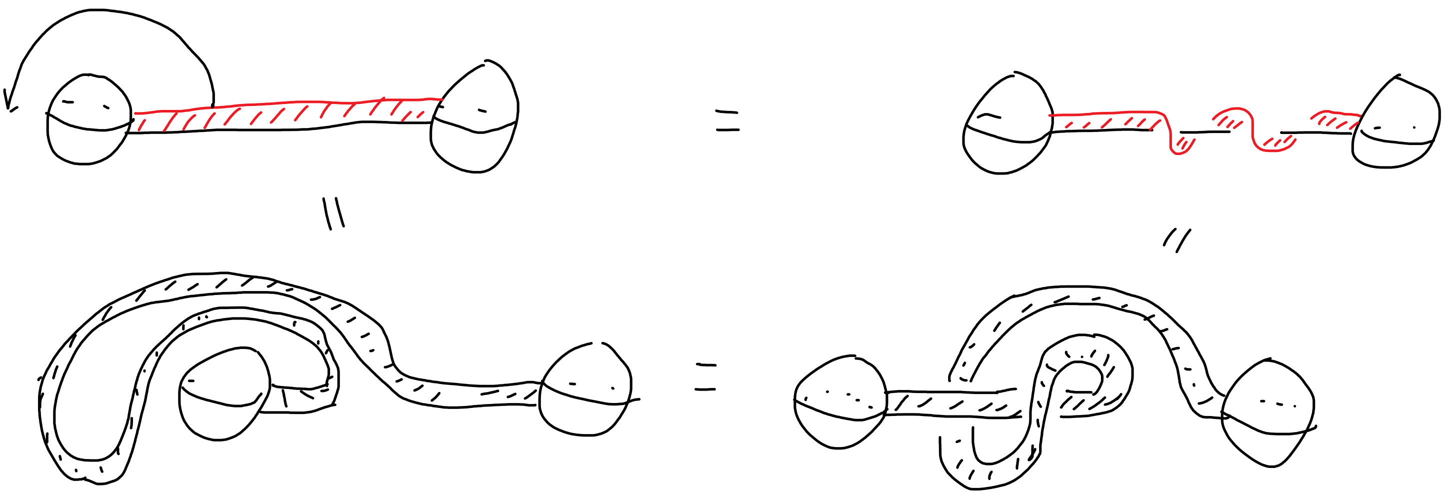


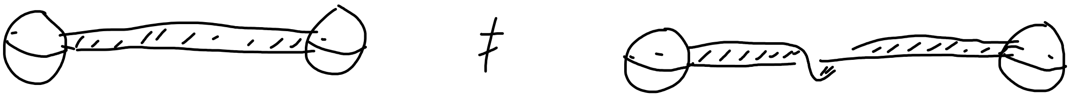
$L(2,1)=\mathbb{RP}^3 \quad \# \quad S^3 \quad \# \quad S^2\times S^2 \quad = S^2\times S^2 \# \mathbb{RP}^3$

SHEET 3

EX 1

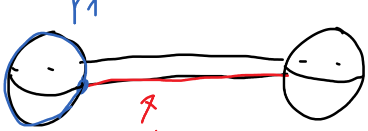






$$\{\text{framings of } S^1\times pt \text{ in } S^1\times S^2\} \xleftrightarrow{1:1} \pi_1(\mathrm{Diff}(S^2), id)$$

$$\overset{\text{SMALE'S THM}}{=} \pi_1(SO(3), id) = \pi_1(\mathbb{RP}^3) = \mathbb{Z}_2$$

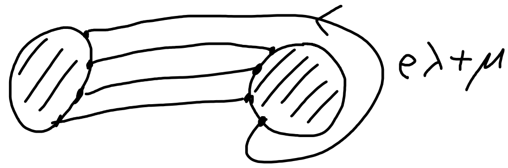


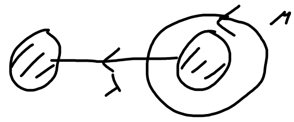
$= S^4$

↑ cancelling 1-/2-handle

all 4 diag $= S^4$

$\boxed{\text{EX 2}}$ (a)

CLAIM: $O^{e} \overset{!}{=} -L(e,1) =$ 



Proof:

Attaching a 2-handle to D^4 along K with framing e, has the following effect on $\partial h_0 = S^3$:

Remove: $\nu K \overset{\text{framing}}{\cong} S^1 \times D^2$

Re-glue $V_0 = D^2 \times S^1$

$$D^2 \times S^1 = V_0 \longrightarrow S^3 \setminus \mathring{\nu K}$$

$$\partial D^2 \times pt = \mu_0 \longmapsto \lambda = e\mu_K + \lambda_{SEIFERT}$$

$$pt \times S^1 = \lambda_0 \longmapsto \mu = \mu_K$$

For $K =$ unknot $\Rightarrow S^3 \setminus \mathring{\nu K} \cong S^1 \times D^2 = V_1$

$\Rightarrow \partial(O^e) =$ lens space $= V_0 \cup V_1$

$$V_0 = D^2 \times S^1 \quad \cup \quad S^3 \setminus \mathring{\nu K} = V_1$$

$$\mu_0 \longmapsto e\mu_K + \lambda_{SEIFERT} = e\lambda_1 + \mu_1$$

$$\lambda_0 \longmapsto \mu_K = \lambda_1$$

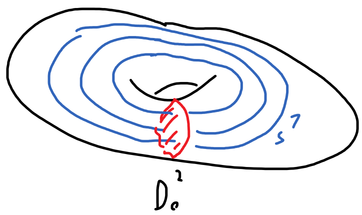
$\Rightarrow \partial(O^e) = -L(e,1)$

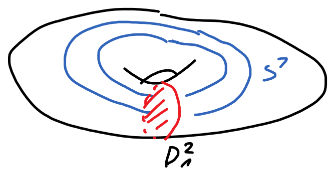
If M is an oriented mfd.

Def $-M$ to be M with the other orientation $(\bar{M})$

(b) $L(e,1) = V_0 \cup V_1$

$$\mu_0 \longmapsto \mu_1 - e\lambda_1$$
$$\lambda_0 \longmapsto \lambda_1$$





$$S^1 \times \underset{\partial D^2}{\underset{\cap}{pt}} = \lambda_0 \longmapsto \lambda_1 = S^1 \times \underset{\partial D^2}{\underset{\cap}{pt}}$$

$$S^1 \hookrightarrow L(e,1) \longrightarrow D_0^2 \cup D_1^2 = S^2$$

Remark: $S^2 \not\subset L(e,1)$

For $e=1$, we get Hopf fibration $S^1 \longrightarrow S^3 \longrightarrow S^2$

$$\left(S^3/\mathbb{Z}_p = L(p,q) \right)$$

(c) $H_1(L(e,1)) = \mathbb{Z}_e$

$\Rightarrow$ for $e \neq 0, \pm 1$: $L(e,1) \neq \#_k S^1 \times S^2$

& $L(0,1) = S^1 \times S^2$ & $L(1,\pm 1) = S^3$

$\Rightarrow$ only $\bigcirc^0$ & $\bigcirc^{\pm 1}$ represent closed 4-mfds

$\bigcirc^0$

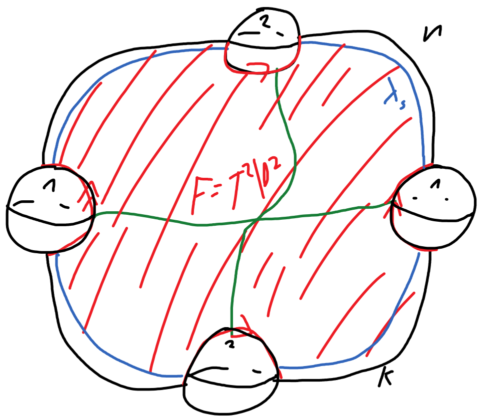
$W_2 = S^2 \times D^2$

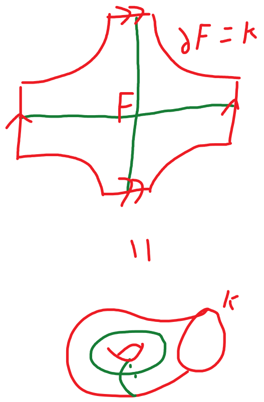
$W = S^2 \times D^2 \cup D^3 \times S^1 \overset{\text{Lemma}}{=} S^4$

$$\boxed{\bigcirc^{\pm 1} = \pm \mathbb{CP}^2}$$ (see last sheet)

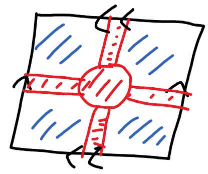
$\boxed{\text{Ex 5}}$

(a)





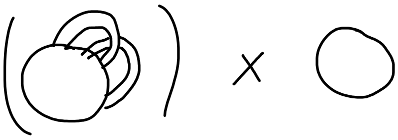
$k = \partial F \implies k = 0 \in H_1(\partial W_1)$



$H_2(W) = \langle T^2 = \text{core of } h_2 \cup F \rangle_{\mathbb{Z}}$

$Q_W = (n) \implies$ Euler number $= n$

(b) $W_1 = h_0 \cup h_1^1 \cup h_1^2 \cong S^1 \times D^3 \natural S^1 \times D^3 \cong (T^2 \setminus D^2) \times D^2$

$\Big[(T^2 \setminus D^2) \times D^2 = \Big($  $\Big)$

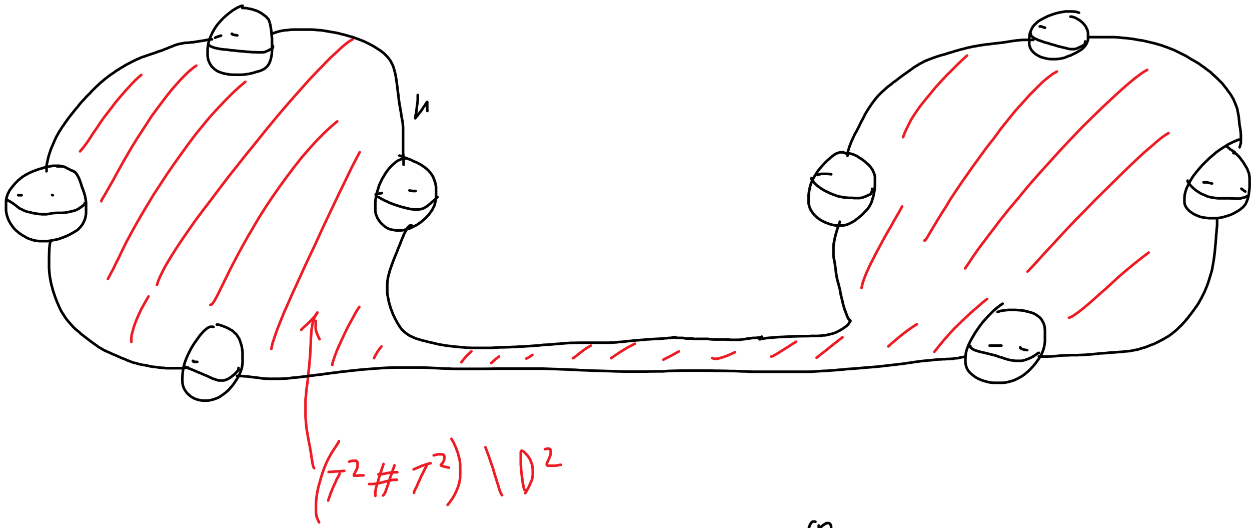
$= h_0^{(2)} \cup h_1^{(2)} \cup h_1^{(2)} \times D^2$

$= h_0^{(4)} \cup h_1^{(4)} \cup h_1^{(4)} = W_1 \Big]$

$W = W_1 \cup h_2$ attached along $k = \partial(T^2 \setminus D^2)$

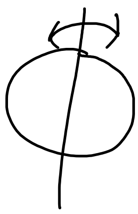
$\implies W = D^2$-bundle over T^2

$$\left[\begin{array}{ccccc} D^2 & \hookrightarrow & W & \longrightarrow & T^2 \\ & & \parallel & & \parallel \\ D^2 & \longrightarrow & W_1 = D^2 \times (T^2 \setminus D^2) & \longrightarrow & T^2 \setminus D^2 \\ & & + & & + \\ \text{co-core of } h_2 = D^2 & \longrightarrow & h_2 & \longrightarrow & D^2 = \text{core of } h_2 \\ & & \parallel & & \\ & & D^2 \times D^2 & & \end{array}\right]$$



$\boxed{EX 4}$ M <u>REVERSIBLE</u> $\Leftrightarrow$ $M \overset{C^\infty}{\cong} -M$

(a) $S^n \cong -S^n$ (reflecting along hyperplane in $\mathbb{R}^{n+1}$)



$S^1 \times M \cong -(S^1 \times M)$ (reflect S^1)

(b) <u>CLAIM:</u> $\mathbb{C}P^2 \not\cong -\mathbb{C}P^2$ $\qquad \mathbb{C}P^2 = h_0 \cup h_2 \cup h_4$

$H_2(\mathbb{C}P^2) = \mathbb{Z}$

$Q_{\pm\mathbb{C}P^2} = e(h_0 \cup h_2) = \pm 1 \quad \Rightarrow \quad \mathbb{C}P^2 \not\cong -\mathbb{C}P^2$

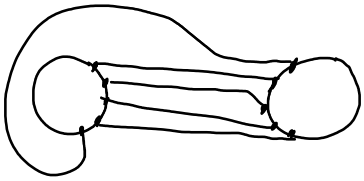
$\left[f: W_1 \xrightarrow[\cong]{\text{ori. pres.}} W_2 \right.$

$\left. F_1 \cdot F_2 = f(F_1) \cdot f(F_2) \quad \Rightarrow \quad Q_{W_1} = Q_{W_2} \right]$

(b) $M = H_1 \cup_{\Sigma} H_2$

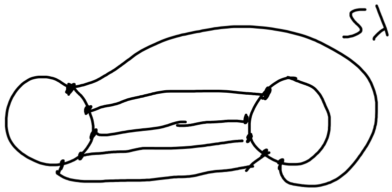
$-M = -H_1 \cup_{-\Sigma} -H_2$

$\Rightarrow$ reflect the Heegaard diagram.

Ex: $L(5,1) =$ 

$-5\lambda+\mu$

$5\lambda+\mu$

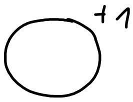
$-L(5,1) =$  $= L(-5,1) = L(5,-1)$

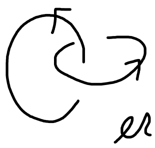
(c) $-W = -h_0 \cup -h_1^i \cup -h_2^i \cup -h_3^i \cup -h_4$

$\Rightarrow K_{-W} = -K_W$ (reflect the Kirby diagram)

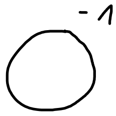
(d) $S^4 = \emptyset = -S^4$

$S^1 \times S^3 =$  $= -(S^1 \times S^3)$

$\mathbb{C}P^2 =$  $=$ 

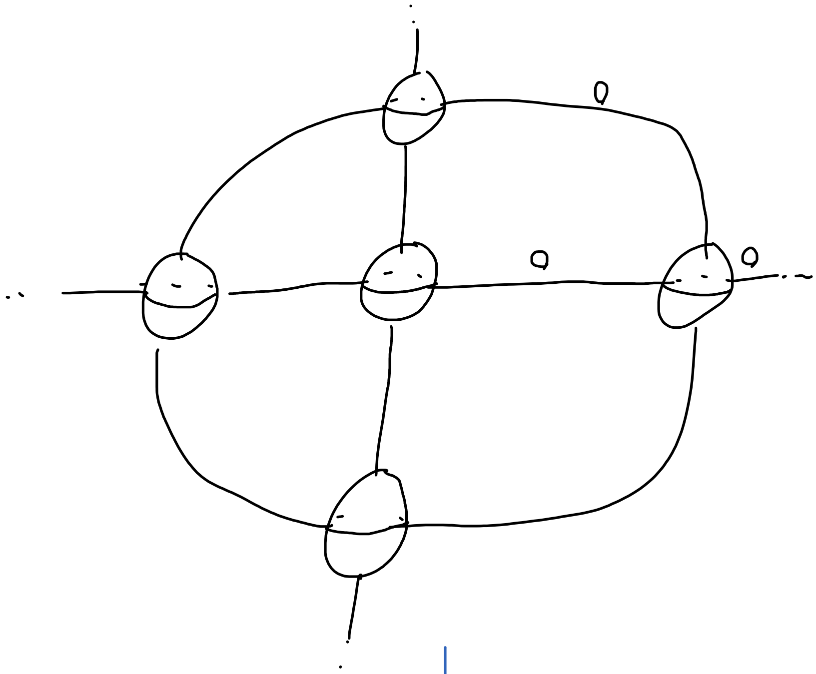


$lk = +1$

$\Rightarrow -\mathbb{C}P^2 =$  $=$ 

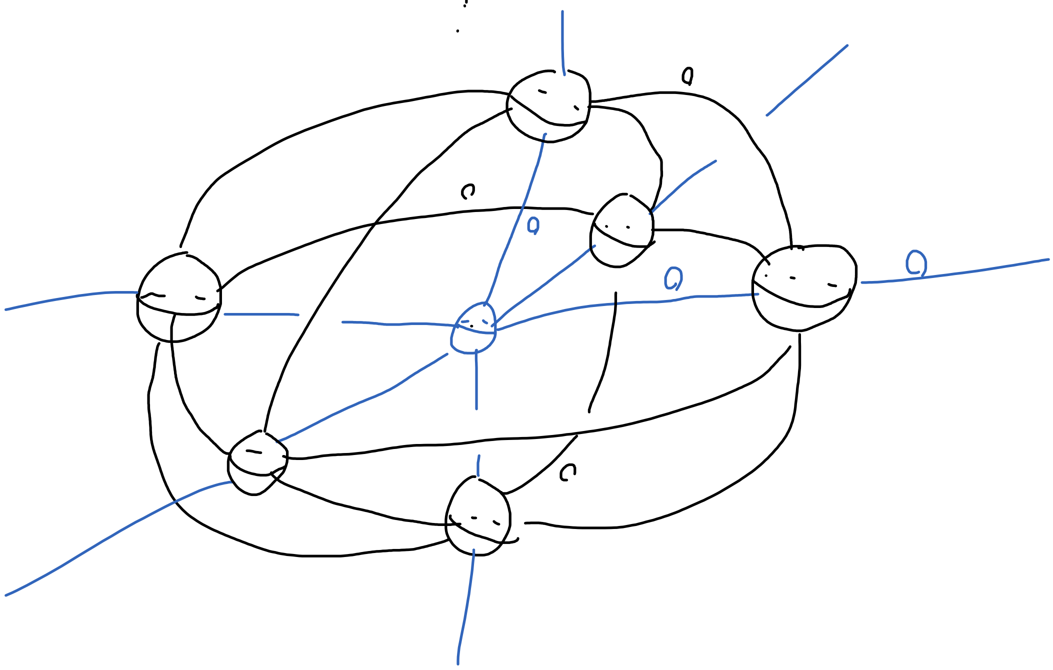
$lk = -1$

$I \times T^3 =$

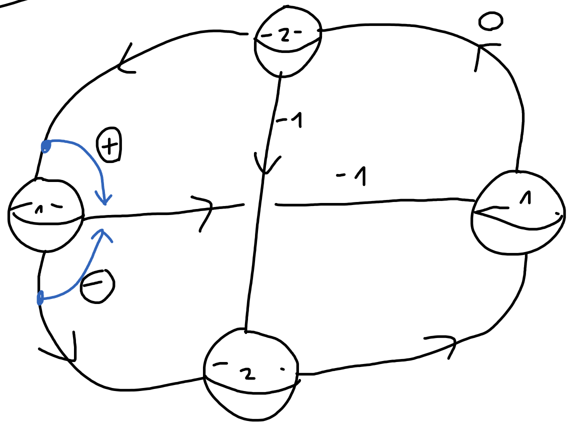


$h_k \times h_\ell = h_{k+\ell}$

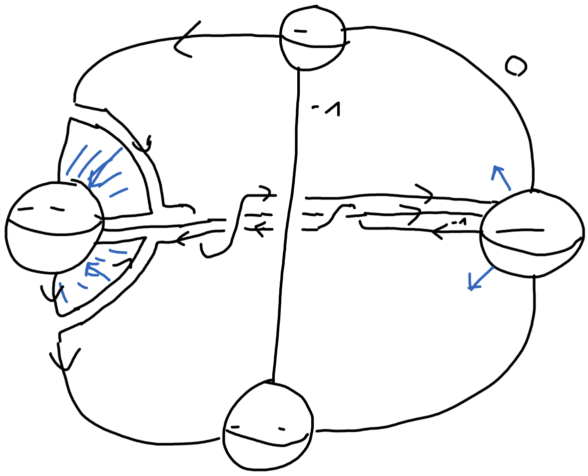
$S^1 \times T^3$



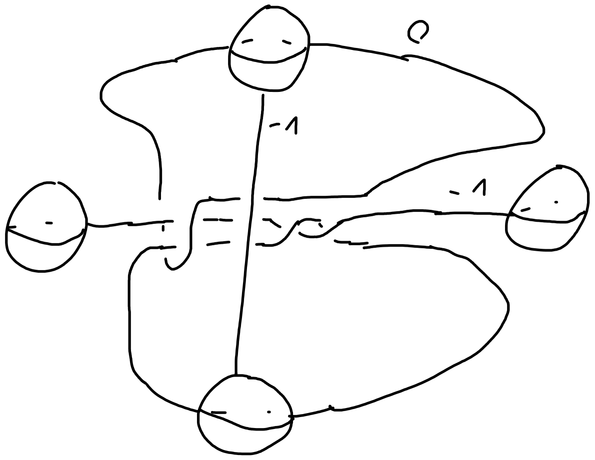
SHEET 4: EXS:



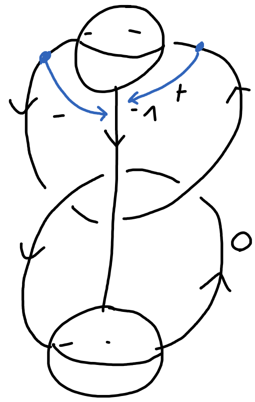
$\times 2$
2.H.S.
$\cong$



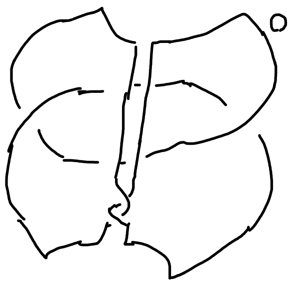
ISOTOPY
$\cong$



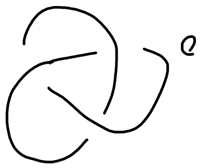
CANCEL.
$\cong$



2x
2.HS & CANC
$\cong$

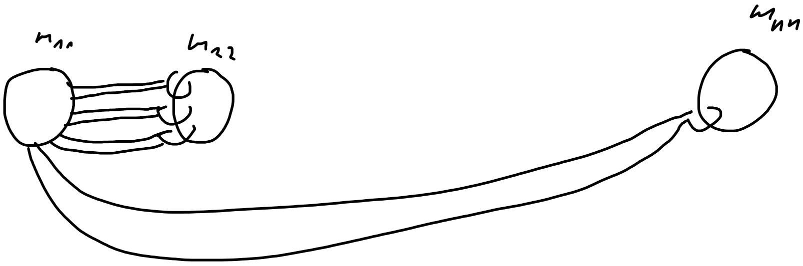


ISOTOPY
$\cong$

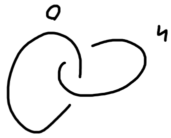


$\boxed{\text{Ex 4}}$ (a) $Q = (M_{ij})$

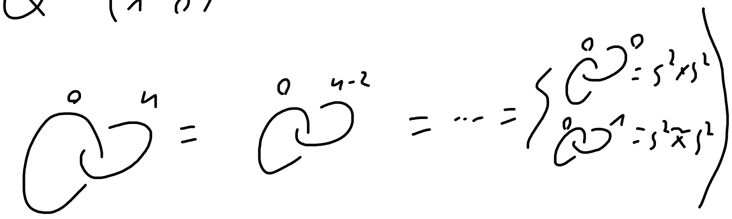
W = Kirby diagram of m_{ii}-framed unknots K_i s.t. $lk(K_i, K_j) = M_{ij}$



$W = h_0 \cup \{h_2^i\}$ $\Rightarrow$ $\pi_1 = 1$ compact $\partial W \neq \emptyset$ smooth

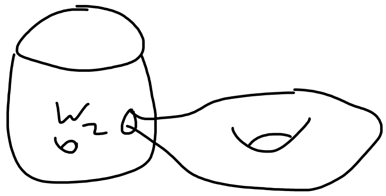
(b) $W :=$  $\quad Q = \begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix}$

(S^2-bundles over S^2

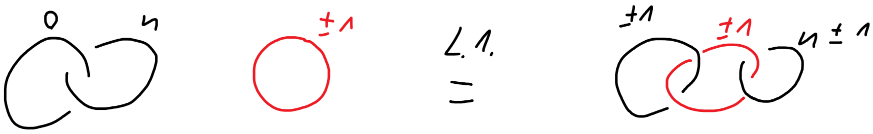


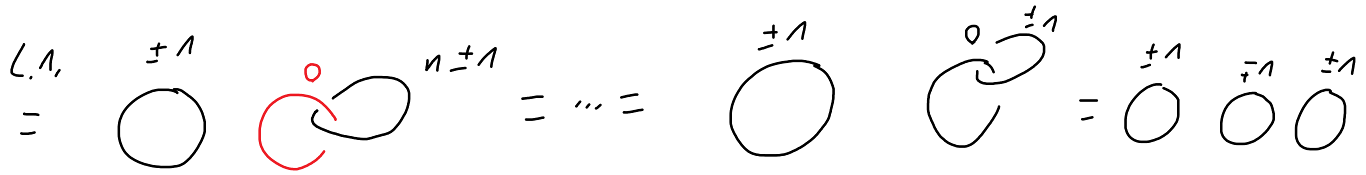
* To show: $\partial\Big($  $\Big) = S^3$

$\partial(W_2) = \partial(W_2 \# \pm \mathbb{C}P^2)$

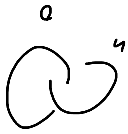


Proof →





represents $\pm\mathbb{C}P^2 \# \mp\mathbb{C}P^2 \# \pm\mathbb{C}P^2$

(c) *  $= S^2$-bundle over S^2 $\underset{\text{via 2-HS}}{\Rightarrow}$ 

2-HS ≙ base change in H_2

* Algebra:

$$Q(x,y) = x^t M y \quad \text{in basis}$$

Let C basis transformation

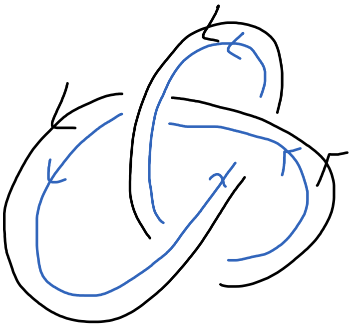
$$Q(Cx, Cy) = (Cx)^t M (Cy) = x^t \underbrace{C^t M C}_{\tilde{M}} y$$

Here: $M = \begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix}$ $\qquad$ GOAL: $C = \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}$

$$\begin{pmatrix} n\pm 2 & 1 \\ 1 & 0 \end{pmatrix} = \tilde{M} = C^t M C = \begin{pmatrix} n\pm 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix} = \begin{pmatrix} n \pm 2 & 1 \\ 1 & 0 \end{pmatrix}$$

Ex 1:

(a)

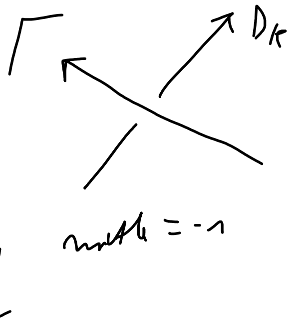


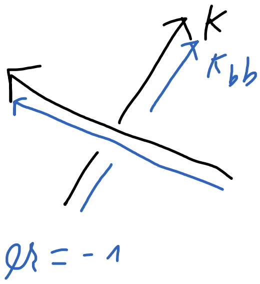
D_K

$lk(K, K_{bb}) = -3$

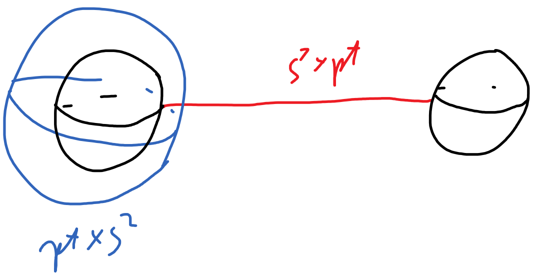
$\text{writhe}(D_K) = -3$

CLAIM: $lk(K, K_{bb}) = \text{writhe}(D_K) = \#$ self crossings of D_K (with signs)



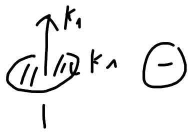


(b)



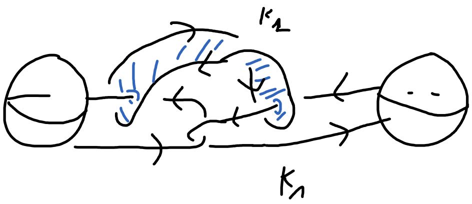
$= S^2 \times S^2$

$[K] = K \cdot (pt \times S^2) \cdot [S^2 \times pt] \in \underbrace{H_1(S^2 \times S^2)}_{\cong \mathbb{Z}_{[S^2 \times pt]}}$

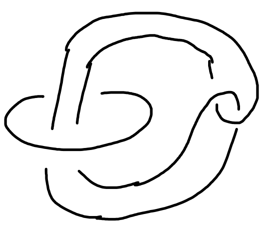


$K_i \cdot (pt \times S^2) = 0 \implies [K_i] = 0 \in H_1(S^2 \times S^2)$

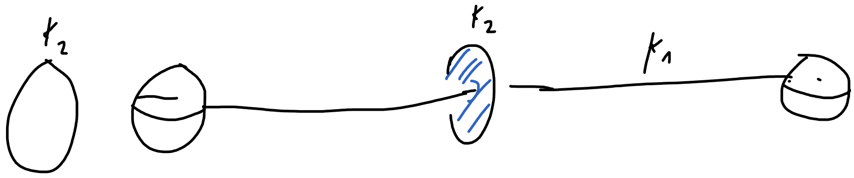
$lk(K_1, K_2) = K_1 \cdot F_2 = -2$



$lk(K_1, K_2) = K_1 \cdot F_2 = 0$



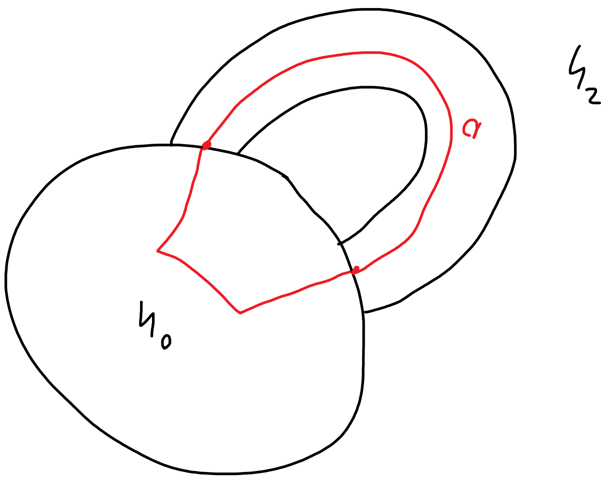
(c)

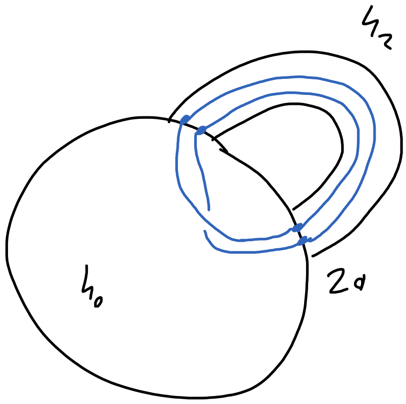


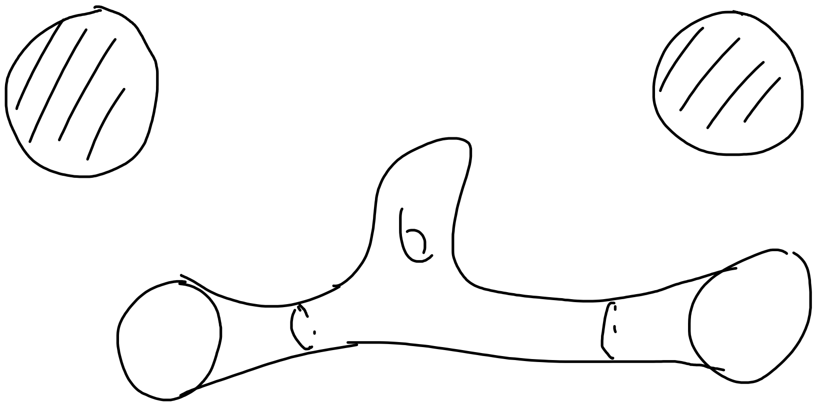
K_1 <u>NOT</u> nullhomotopic $\Rightarrow$ lk not well-def

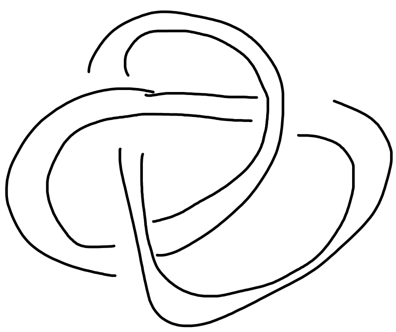
$$[K_1] = lk \cdot [\mu_2] \in H_1(M^3 \setminus \nu K_2) = H_1(M) \oplus \mathbb{Z}\langle \mu_2 \rangle$$

<u>Ex 2:</u>







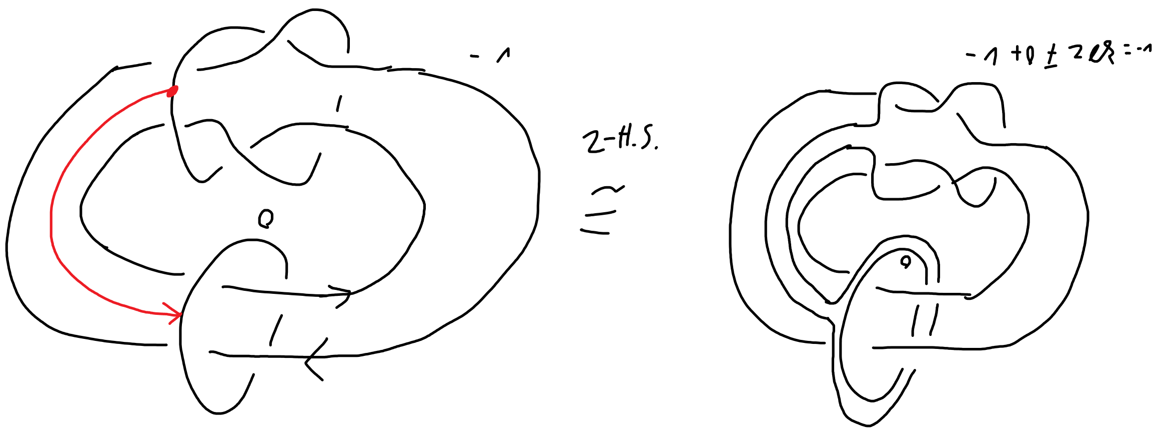


SHEET 5

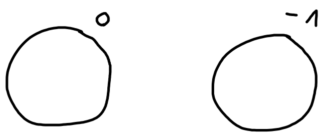
	4-MFDS W_2	closed 4-mfds W	∂W_2	M^3
1-HANDLE SLIDES	✓	✓	✓	✓
2-HANDLE SLIDES	✓	✓	✓	✓
1-2-HANDLE CANCEL.	✓	✓	✓	✓
2-3-HANDLE CANCEL.	X 4 $O^0 = S^2\times D^2$	✓	$\# O^0 = S^2\times S^2$ X	X
BLOW UPS / DOWNS	X O^{+1} # $\mathbb{CP}^2$	X	✓	✓
ROLFSEN TWISTS	N.D.	N.D.	N.D.	✓
SLAM DUNKS	N.D.	N.D.	N.D.	✓

$\#_k S^2\times S^2 =$

EX1



ISOTOPY ≅



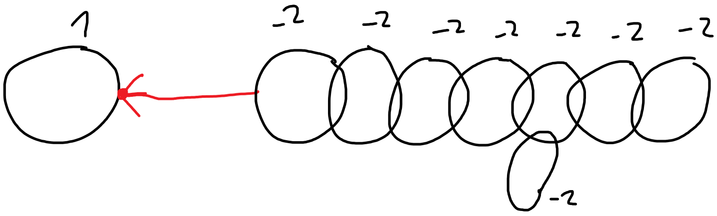
$W_2 = S^2\times D^2 \cup D^2$-handle on S^1, $e = -1$

$\partial W_2 = M = S^2\times S^1 \# S^3 = S^2\times S^1$

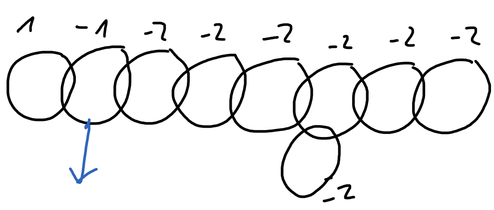
$W = -\mathbb{CP}^2$

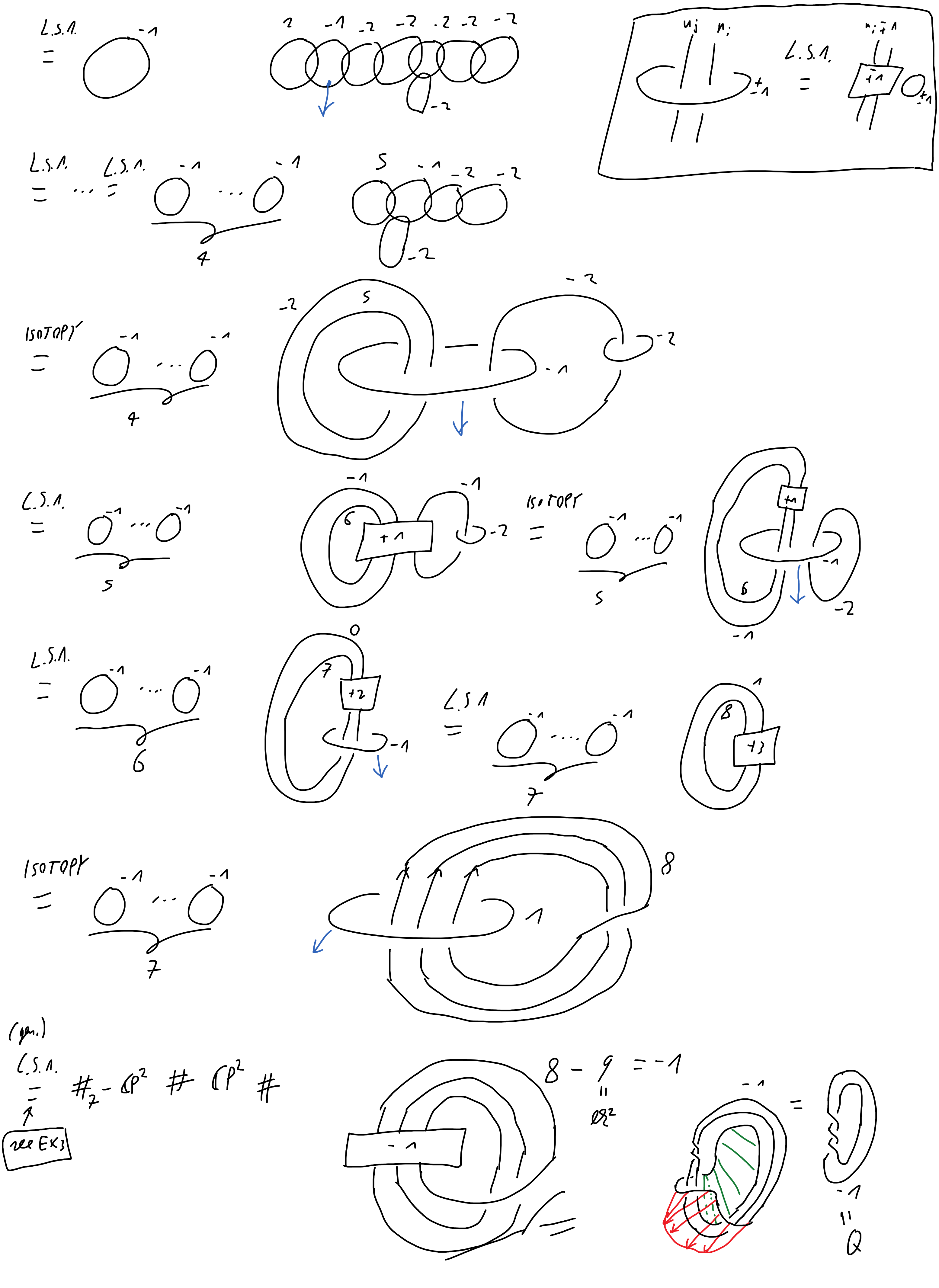
EX2

$P \# \mathbb{CP}^2 =$

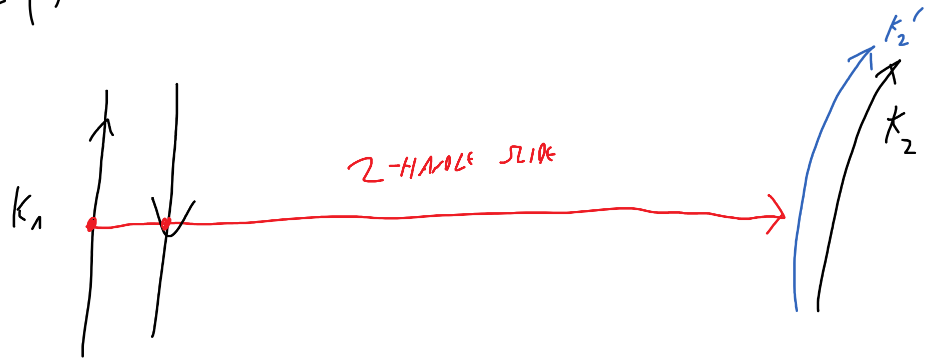


L.S.A. =

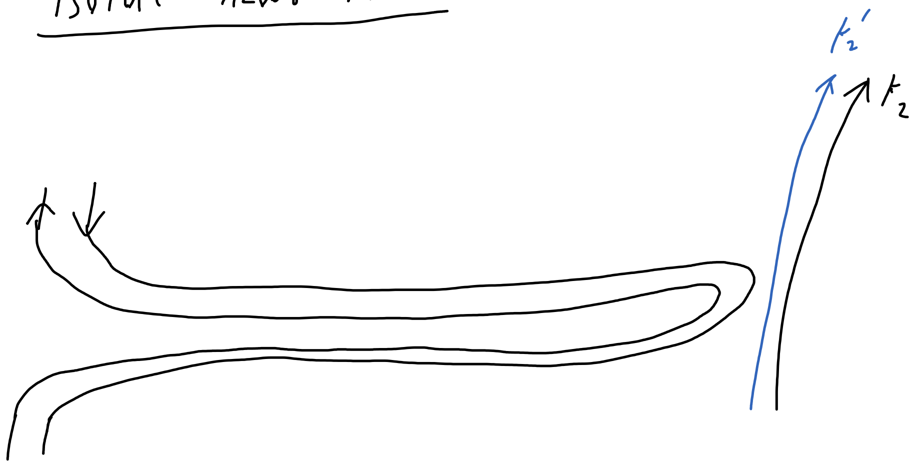



L.S.A.
ISOTOPY
(gen.)
#7 -CP² # CP² #
see EX3
8 - 9 = -1
Q

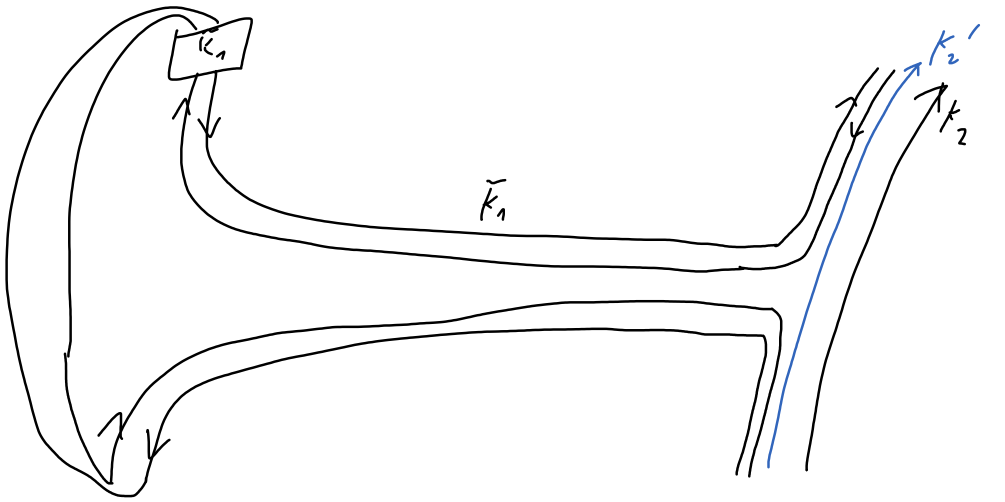
EX 3 (d)



① ISOTOPE ALONG BAND:

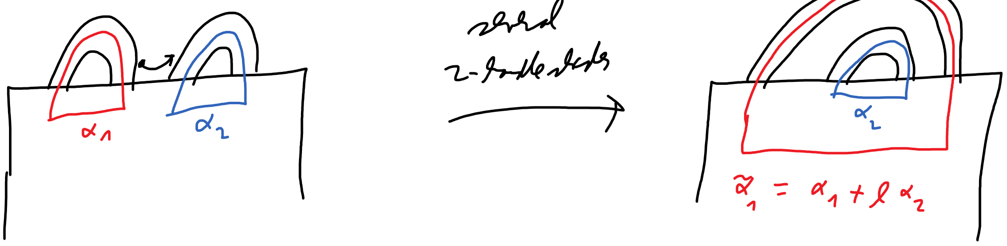


② ISOTOPY "OVER" $\{pt\} \times D^2 \subset D^2 \times D^2 = h_2(K_2) \triangleq \#$ with K_2'



W = 2-handlebody

$\alpha_1, \dots, \alpha_n$ basis of $H_2(W)$ given by K_i'



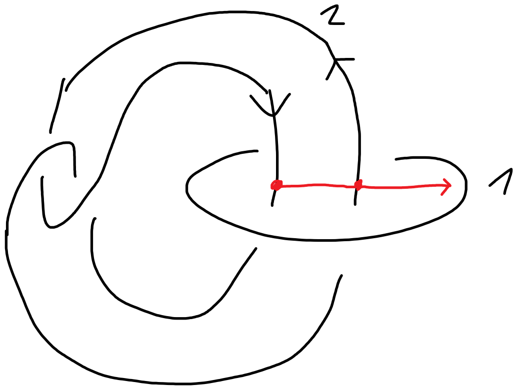
$\tilde{\alpha}_1 = \alpha_1 + \ell \alpha_2$

where ℓ = # strands of K_1 counted with signs framing: $\tilde{n}_1 = (\alpha_1 + l\alpha_2)\cdot(\alpha_1 + l\alpha_2)$

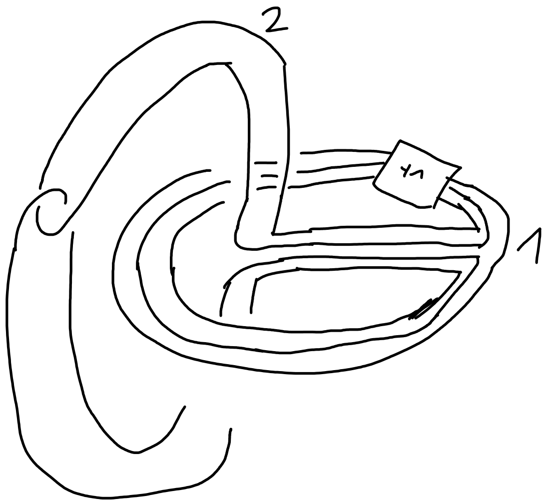
$$= \alpha_1\cdot\alpha_1 + 2l\,\alpha_1\cdot\alpha_2 + l^2\,\alpha_2\cdot\alpha_2$$

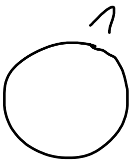
$$= n_1 + 2l\,\mathrm{lk}(K_1,K_2) + l^2\,n_2$$

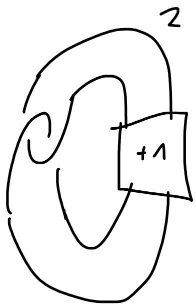
Ex:



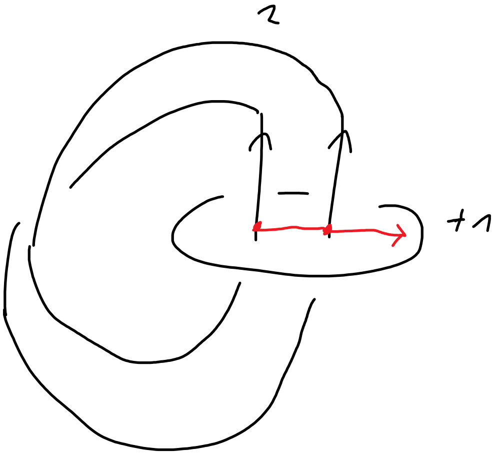
$\cong$





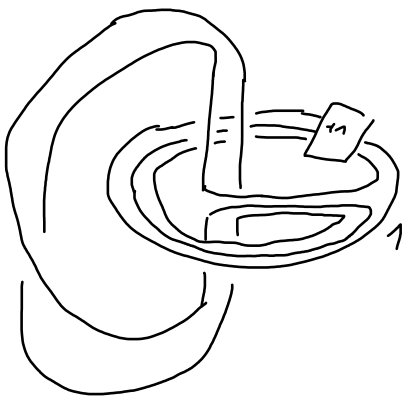


Ex.

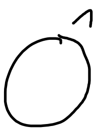


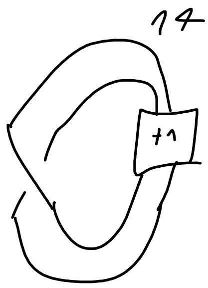
$\cong$

$2+8+4=14$

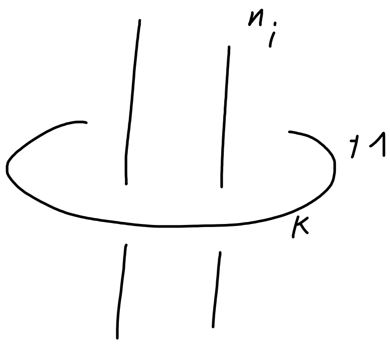


$\cong$

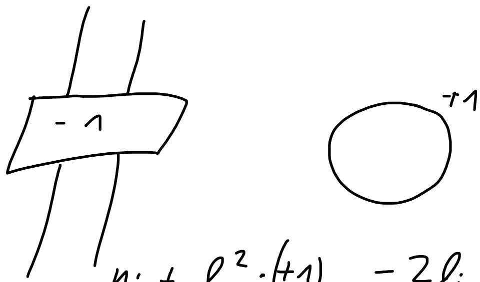




(b)



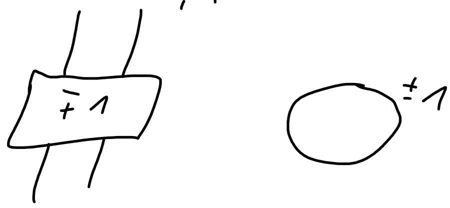
$=$



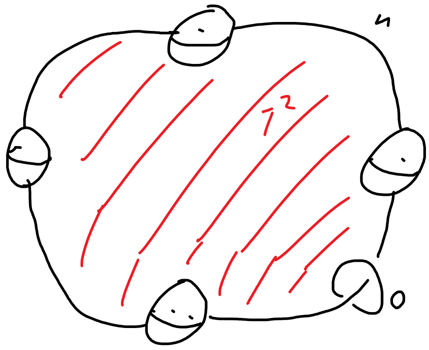
$n_i + l_i^2\cdot(+1) - 2l_i\, lk(K_i,K)$

If $l_i = lk$

$=$



EX4 $D\left(D^2\text{-bundle over } T^2 \text{ with } e=n\right) =$

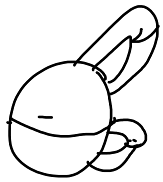


Also same as THM 5.3.

EX5 (a) $\pi_1(M) = \pi_1(M_2) = \langle h_1^1,\dots,h_1^i \mid h_2^1,\dots,h_2^j\rangle$

(b) * Let $G = \langle g_1,\dots,g_k \mid r_1,\dots,r_\ell\rangle$

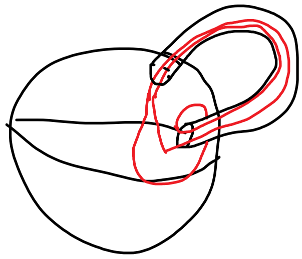
$M_1 = h_0 \cup h_1^1 \cup\dots\cup h_1^k$



$\pi_1(M_1) = \langle g_1,\dots,g_k\rangle$

* $\dim(M) = n \geq 4 \Rightarrow \dim(\partial M_1) \geq 3$

represent r_i by embedded curves in ∂M_1



$M_2 = M_1 \cup h_2^1 \cup \dots \cup h_2^\ell$ attached along the r_i

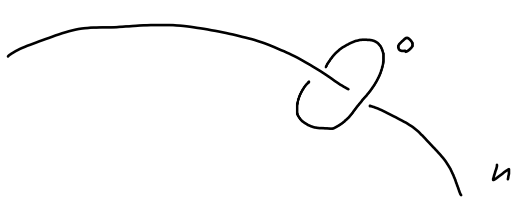
$\Rightarrow \pi_1(M_2) = G$

CLAIM: $\pi_1(DM_2) = G$

- $n \geq 5$ $\quad DM_2 = M_2 \cup \{\text{index} \geq 3\}$

- $n = 4$ $\quad DM_2 = M_2 \cup h_2^* \cup h_3^* \cup h_4^*$

  h_2^* attached along b_2



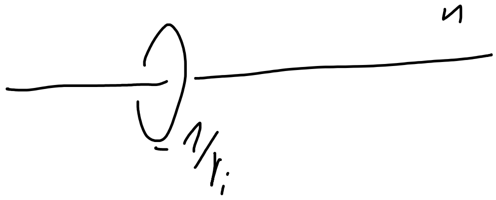
$\bigcirc^\circ$ is nullhomotopic in M_2

SHEET 6

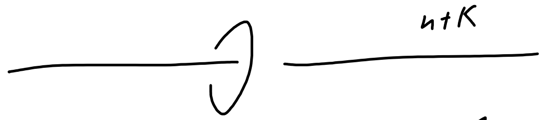
Ex 2

(a) $\partial(D^2\text{-bundle over } \Sigma_g \text{ with } e=n) = M(g, n\,;\, 0, \dots, 0)$

(b)



$\overset{k\text{-fold}}{\underset{RT}{=}}$



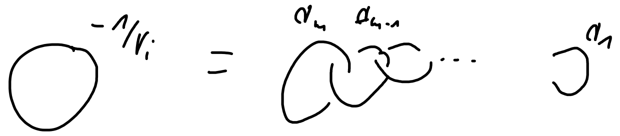
$$\frac{1}{k - \frac{1}{1/r_i}} = \frac{1}{k - r_i}$$

(c) use algorithm from lecture:

write $-1/r_i = a_m - \cfrac{1}{a_{m-1} - \dots - \cfrac{1}{a_1}}$ (Euclidean algorithm)

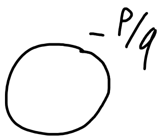
$a_i \in \mathbb{Z}$

$\Rightarrow$



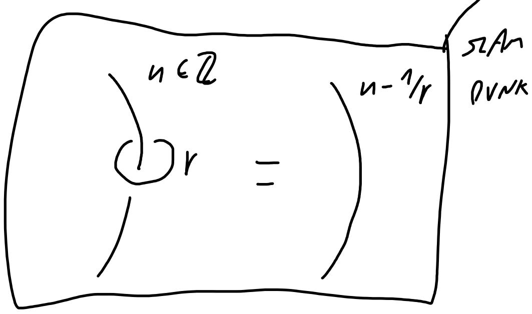
$\Rightarrow$ get integer surgery diag., seen as a Kirby diag. of W

(d) $L(p,q) =$



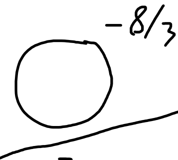
$\cong$

$= M(0, 0\,;\, -p/q)$

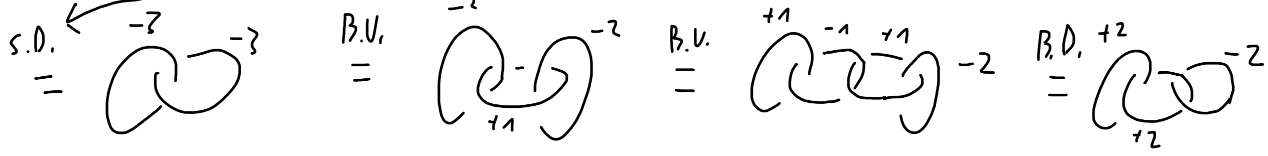


slam dunk

(e) $L(8,3) =$

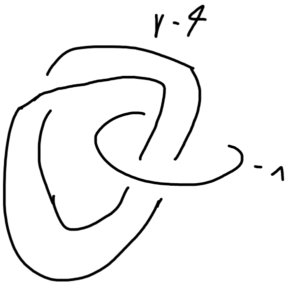


$$\boxed{-8/3 = -3 - \frac{1}{-3}}$$

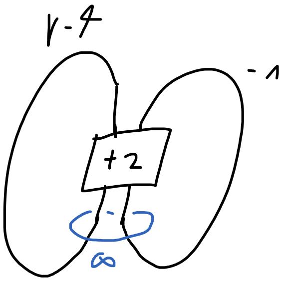


(1)

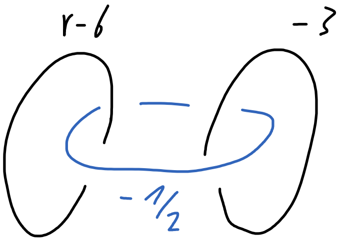
R.T.
$\cong$



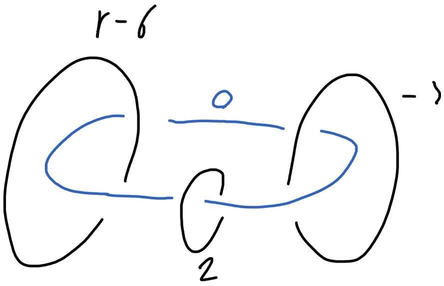
ISOTOPY
$\cong$



(-2)-~~fold~~
RT
$\cong$



S.D.
$\cong$



$= M\left(0, 0; -\frac{1}{2}, \frac{1}{3}, -\frac{1}{r-6}\right)$

$\boxed{\text{EX 3}}$ $\boxed{\text{PROP R}}$ (GABAI)

$$S^3_r(K) = S^1 \times S^2 \Rightarrow K = O \;\&\; r = 0$$

(a) $M = h_0 \cup h_1^j \cup h_2 \cup h_4$

$$H_*(M) = H_*(S^4) \Rightarrow \chi(M) = \chi(S^4) = 2$$

$$\Rightarrow 2 = \chi(M) = 1 - j + 1 + 1 \Rightarrow j = 1$$

$$\Rightarrow M_1 = S^1 \times D^3$$

dual hand. decomp $\Rightarrow M = h_0 \cup h_2 \cup h_3 \cup h_4$

$$\Rightarrow \partial M_2 = S^1 \times S^2$$

PROP R $\Rightarrow \partial M_2 = O^0$

Cerf

$\Rightarrow M = S^4$

(b) GENERALIZED PROP. R CONJECTURE (OPEN) (*)

$$L = L_1 \cup \dots \cup L_n \quad \text{s.t.} \quad L(n_i) = \#_n S^1 \times S^2 \qquad n_i \in \mathbb{Z}$$

$$\overset{?}{\Rightarrow} L = \underbrace{\bigcirc^0 \cdots \bigcirc^0}_{n\text{-comp}} \quad \text{after 2-handle slides.}$$

CLAIM: If (*) is true then:

$$H_*(M) = H_*(S^4) \ \& \ M \text{ has NO 3-handles} \Rightarrow M \overset{C^\infty}{\cong} S^4$$

Proof: $M = h_0 \cup h_1^k \cup h_2^m \cup h_4$

$$2 = \chi(S^4) = \chi(M) = 1 - k + m + 1 \quad \Leftrightarrow \quad k = m =: n$$

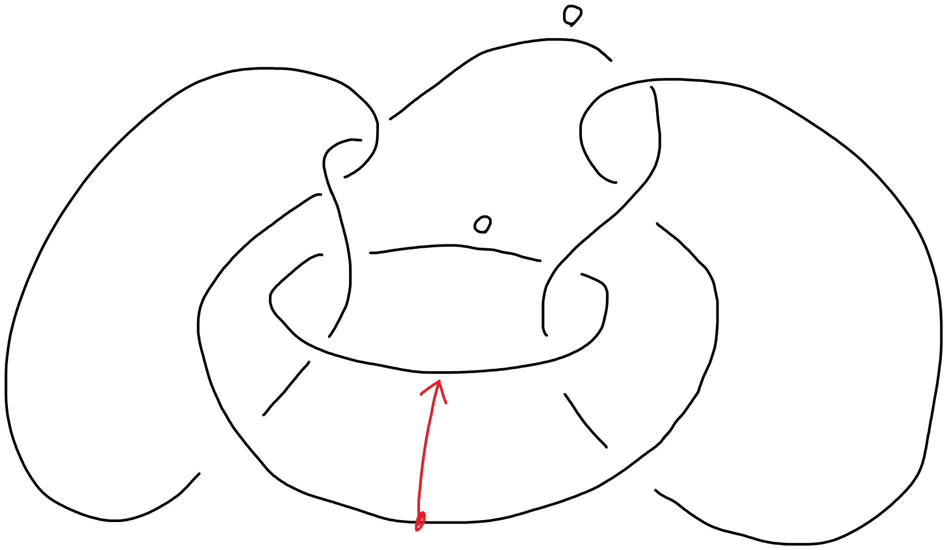
$\rightarrow$ dual handle decomp $\quad M = h_0 \cup h_2^n \cup \underbrace{h_3^n \cup h_4}_{\natural_n S^1 \times D^3}$

$$\Rightarrow \partial M_2 = \#_n S^1 \times S^2$$

$$\overset{(*)}{\Rightarrow} M_2 \overset{C^\infty}{\cong} \bigcirc^0 \cdots \bigcirc^0 \quad \text{(after 2-handle slides)}$$

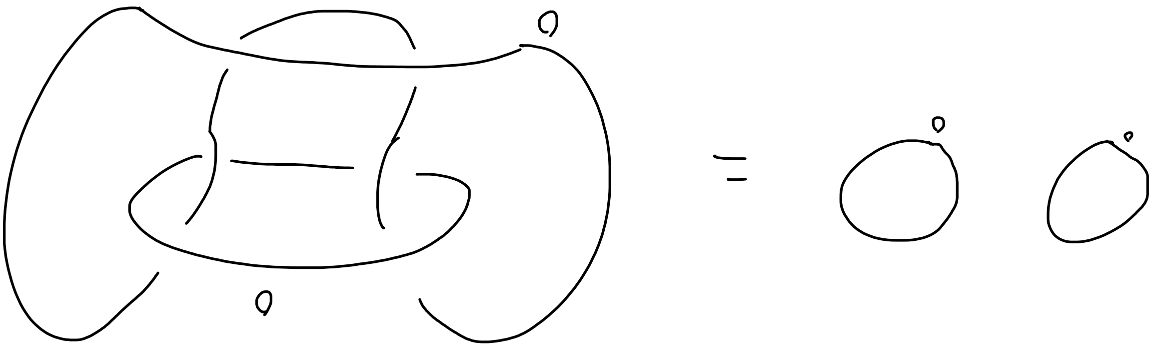
$$\overset{\text{cancellation}}{\Rightarrow} M \overset{C^\infty}{\cong} S^4$$

(c)



$= \#_2 S^2 \times S^2$

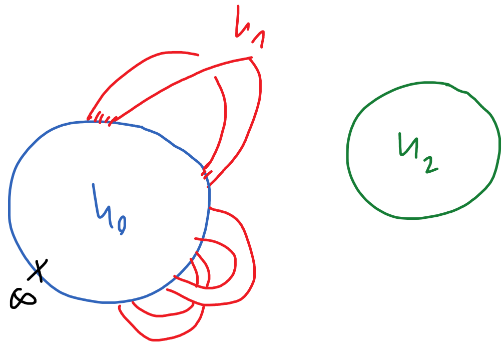
=



(d) compute homology

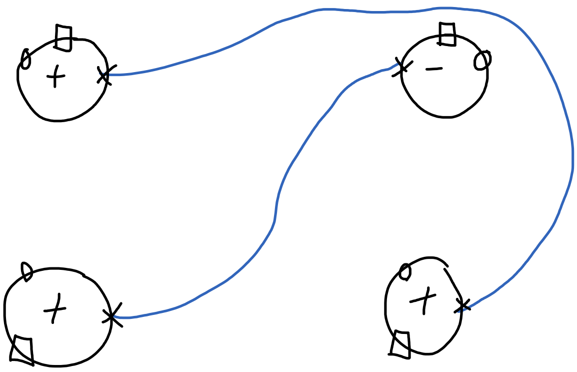
(e) homology sphere with $\pi_1 = 1$ $\overset{?}{\Rightarrow}$ for a diagram without 1-handles + gen. prop R conj.

$\Rightarrow$ S4PC

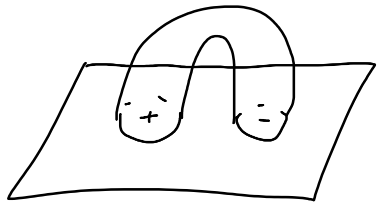


$= \mathbb{R}P^2 \# T^2$

$\partial h_0 \supset \mathbb{R}$

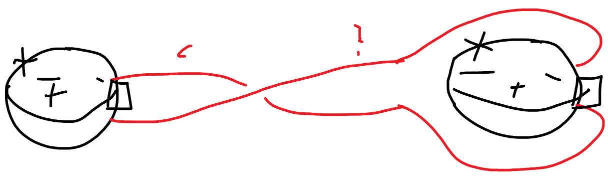


$\hat{=}$



$\hat{=}$ attached 1-handle or. reversing

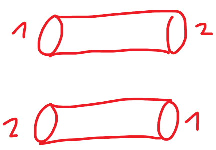
$\hat{=}$ attached 1-handle or. preserving



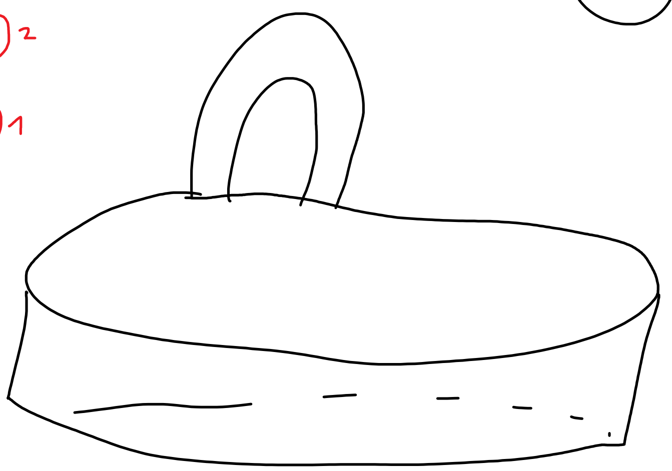
$\hat{=}$ '' reversing

$V \subset \overset{!}{=} S^1 \times D^2$

//



$V \subset \ncong S^1 \times D^2$



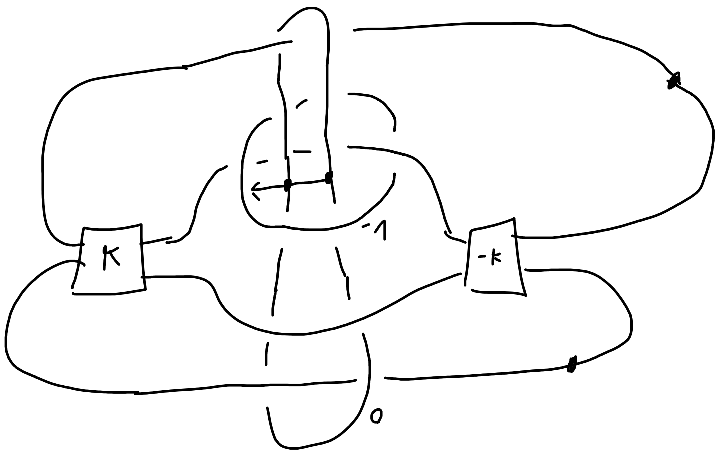
SHEET 7

Ex 3: (a) rotation by 180°

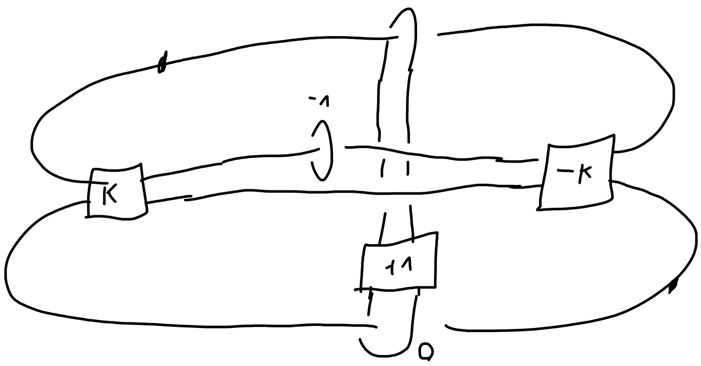
(b) Claim: $H_{0,K} \cong D^4$

Proof:

$H_{0,K}$ =

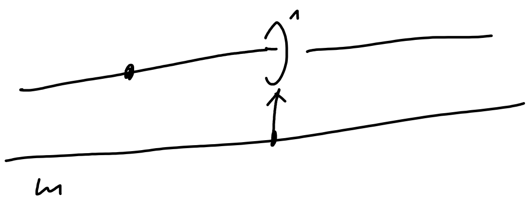


$\cong$

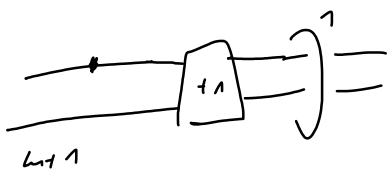


$\overset{\text{cancel}}{=}$ $\emptyset$ $= D^4$

Ex 4:
(a)

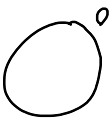


$\overset{\text{handle slide}}{=}$

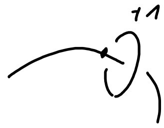


▨

(b) do ex 2 again & keep track of the 0-framed meridian:



(c) $\boxed{K} \sqcup O^0 \overset{\text{always}}{\cong}$ cancel 2-3-handle pair



$\underset{\text{ex}}{\overset{\text{in this}}{\cong}}$ cancel 2-3-handle pair

KIRBY DIAG

$$w = W_2 \cup h_3^i \cup h_4$$

$$h_1^i \cup h_0$$

$$\cong \natural_i S^1 \times D^3$$

$$\Rightarrow \partial W_2 = \partial \natural_i S^1 \times D^3 = \#_i S^1 \times S^2$$